491 Knowledge Representation

Tutorial Exercise

Nonmonotonic consequence relations

Question 1 Look again at the earlier exercise sheet *Consequence relations*. Solutions to those exercises were handed out. Many of those questions did not assume that the consequence relations referred to were monotonic.

Question 2

Show that the following hold for any supraclassical cumulative operator Cn.
(Cn is cumulative when $A \subseteq B \subseteq \mathrm{Cn}(A)$ implies $\mathrm{Cn}(B) = \mathrm{Cn}(A)$.)

i) $\mathrm{Th}(\mathrm{Cn}(A)) = \mathrm{Cn}(A)$

ii) If $B \subseteq \mathrm{Cn}(A)$ then $\mathrm{Th}(B) \subseteq \mathrm{Cn}(A)$.
(Or equivalently, if $B \subseteq \mathrm{Cn}(A)$ and $C \subseteq \mathrm{Th}(B)$ then $C \subseteq \mathrm{Cn}(A)$.)

iii) If $\{canary, \neg(yellow \wedge blue)\} \mid\!\sim yellow$ then $\{canary, \neg(yellow \wedge blue)\} \mid\!\sim \neg blue$
(where $A \mid\!\sim \alpha$ iff $\alpha \in \mathrm{Cn}(A)$).

PS: Make sure you understand why part (ii) can say 'Or equivalently ...'.

Question 3 Check that the following set of default rules

$$\frac{a:}{c} \qquad \frac{:\neg b}{a} \qquad \frac{c:\neg a}{b}$$

provides an example to show that cautious monotony does not hold for (sceptical, cautious) Reiter default logic.

In other words: suppose (D, W) is a Reiter default theory.
Let Cn_D be the consequence operator corresponding to the 'sceptical' or 'cautious' consequences of W under the default rules D: that is, α is in $\mathrm{Cn}_D(W)$ iff α is in the intersection of all the extensions of the default theory (D, W).
Let $W \mid\!\sim_D \alpha$ be shorthand for $\alpha \in \mathrm{Cn}_D(W)$.
Now suppose D is the set of default rules above. Find formulas α and β (and a set of formulas W) such that $W \mid\!\sim_D \alpha$ and $W \mid\!\sim_D \beta$ but $W \cup \{\beta\} \not\mid\!\sim_D \alpha$.
(Construct the extensions.)

Because of the relationship between logic programs and Reiter default theories, you can, if you prefer, consider the following logic program instead:

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a ← not b
c ← a
b ← c, not a
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Question 4

Let $\mathrm{Cn}_D(W)$ and $W \mid\!\sim_D \alpha$ be as in the previous question.
Show each of the following, for any sets of formulas A and B:

i) $A \subseteq \mathrm{Cn}_D(A)$

ii) When $A \subseteq B \subseteq \mathrm{Cn}_D(A)$, if E is an extension of (D, A) then E is an extension of (D, B).

iii) From part (ii), it follows that Cn_D satisfies the following property *'cumulative transitivity'*:

$$\text{If } A \subseteq B \subseteq \mathrm{Cn}_D(A) \text{ then } \mathrm{Cn}_D(B) \subseteq \mathrm{Cn}_D(A)$$

(Part(i) is very easy. Part (ii) is quite hard, but not if you keep a clear head. Part (iii) follows from part (ii) quite easily.)

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SOLUTIONS

Question 1 (Handed out with earlier sheet)

Question 2

i) One half: $\mathrm{Cn}(A) \subseteq \mathrm{Th}(\mathrm{Cn}(A))$ (inclusion Th).
For the other half:
Cn is cumulative (given) so $A \subseteq \mathrm{Cn}(A) \subseteq \mathrm{Cn}(A)$ implies $\mathrm{Cn}(\mathrm{Cn}(A)) \subseteq \mathrm{Cn}(A)$.
Cn is supraclassical (given) so we have:
$\mathrm{Th}(\mathrm{Cn}(A)) \subseteq \mathrm{Cn}(\mathrm{Cn}(A)) \subseteq \mathrm{Cn}(A)$.

ii) $B \subseteq \mathrm{Cn}(A) \;\Rightarrow\; \mathrm{Th}(B) \subseteq \mathrm{Th}(\mathrm{Cn}(A))$ (Th monot.)
$\mathrm{Th}(\mathrm{Cn}(A)) \subseteq \mathrm{Cn}(A)$ (proved above).
So: $B \subseteq \mathrm{Cn}(A) \;\Rightarrow\; \mathrm{Th}(B) \subseteq \mathrm{Th}(\mathrm{Cn}(A)) \subseteq \mathrm{Cn}(A)$.

PS: $B \subseteq \mathrm{Cn}(A) \;\Rightarrow\; \mathrm{Th}(B) \subseteq \mathrm{Cn}(A)$ is equivalently stated as

$$\text{if } B \subseteq \mathrm{Cn}(A) \text{ and } C \subseteq \mathrm{Th}(B) \text{ then } C \subseteq \mathrm{Cn}(A)$$

You might find this easier to read when written like this:

$$A \mathrel{|\!\sim} B \text{ and } B \vdash C \text{ implies } A \mathrel{|\!\sim} C$$

There is nothing deep here.
'If P and Q then R' is equivalent to 'if P then (if Q then R)'.
For any sets A and B, $X \subseteq A \;\Rightarrow\; X \subseteq B$ for any X is equivalent to saying $A \subseteq B$.

iii) This just applies the previous part and inclusion of Cn.
$\{canary,\ \neg(yellow \wedge blue)\} \mathrel{|\!\sim} yellow$ (given).
$\{yellow\} \subseteq \mathrm{Cn}(\{canary,\ \neg(yellow \wedge blue)\})$

$\{canary,\ \neg(yellow \wedge blue)\} \mathrel{|\!\sim} \neg(yellow \wedge blue)$ (because $A \subseteq \mathrm{Cn}(A)$)
$\{\neg(yellow \wedge blue)\} \subseteq \mathrm{Cn}(\{canary,\ \neg(yellow \wedge blue)\})$.

So:
$\{yellow,\ \neg(yellow \wedge blue)\} \subseteq \mathrm{Cn}(\{canary,\ \neg(yellow \wedge blue)\})$

Now just apply part (ii):
$\{\neg blue\} \subseteq \mathrm{Th}(\{yellow,\ \neg(yellow \wedge blue)\})$

So:
$\{\neg blue\} \subseteq \mathrm{Cn}(\{canary,\ \neg(yellow \wedge blue)\})$
$\{canary, \neg(yellow \wedge blue)\} \mathrel{|\!\sim} \neg blue$.

Question 3

Let $D = \{\dfrac{a:}{c}, \dfrac{:\neg b}{a}, \dfrac{c:\neg a}{b}\}$, $\qquad W = \emptyset$.

The default theory $(D, \emptyset)$ has one extension: $\mathrm{Th}(\{a, c\})$.
(And since the extension is unique, $\emptyset \mathrel{|\!\sim}_D c$.)

$(D, \{c\})$ has two extensions: $\mathrm{Th}(\{a, c\})$ and $\mathrm{Th}(\{b, c\})$.
a which was in the unique extension of $(D, \emptyset)$ is not in the intersection of these two.

In terms of the logic program: the original program obtained by translating $(D, \emptyset)$ has one stable model (answer set) $\{a, c\}$. (I found this by trying all the possible interpretations. I couldn't see a quicker way to do it.)
Add $c \leftarrow$. Now we get two stable models (answer sets): $\{a, c\}$ and $\{b, c\}$.

Question 4

Part (i) is easy. A is a subset of every extension of (D, A) by definition, so $A \in \bigcap ext(D, A)$.

Part (ii). (Actually the essence of the argument is in Question 6 of the exercise sheet on default logic. Anyway, here it is again.)

Suppose $A \subseteq B \subseteq \bigcap ext(D, A)$. Suppose $E \in ext(D, A)$. Then by definition $E = \mathrm{Cn}_{D^E}(A)$.

We need to show $E \in ext(D, B)$, i.e., $E = \mathrm{Cn}_{D^E}(B)$. Equivalently $\mathrm{Cn}_{D^E}(A) = \mathrm{Cn}_{D^E}(B)$.
We do it in two separate parts.

Cn_{D^E} is monotonic so $A \subseteq B$ implies $\mathrm{Cn}_{D^E}(A) \subseteq \mathrm{Cn}_{D^E}(B)$.

It just remains to show $\mathrm{Cn}_{D^E}(B) \subseteq \mathrm{Cn}_{D^E}(A)$.
First, notice that $B \subseteq E$ (because $B \subseteq \mathrm{Cn}_D(A)$ (assumed) means that B is a subset of all extensions of (D, A), and E is one such).
Cn_{D^E} is monotonic so we have:
$\mathrm{Cn}_{D^E}(B) \subseteq \mathrm{Cn}_{D^E}(E) = \mathrm{Cn}_{D^E}(\mathrm{Cn}_{D^E}(A)) \subseteq \mathrm{Cn}_{D^E}(A)$.

Given part (ii), part (iii) is also easy. Part (ii) is equivalently stated as

$$A \subseteq B \subseteq \mathrm{Cn}_D(A) \;\Rightarrow\; ext(D, A) \subseteq ext(D, B)$$

So when $A \subseteq B \subseteq \mathrm{Cn}_D(A)$, $\bigcap ext(D, B) \subseteq \bigcap ext(D, A)$.

Or in words: Suppose $A \subseteq B \subseteq \mathrm{Cn}_D(A)$. Suppose $\alpha \in \mathrm{Cn}_D(B)$. Then α is in every extension of (D, B).
Now suppose E is some extension of (D, A). Then E is also an extension of (D, B) by part(ii). But α is in all extensions of (D, B), so α is also in E.