491 KNOWLEDGE REPRESENTATION

Tutorial Exercise

Fixpoint semantics for definite programs
SOLUTIONS

Question 1

(a) Let $IDB = \{\, p \leftarrow q, r\,;\ q \leftarrow s \,\}$ and $EDB = \{\, r,\ s \,\}$.

Clearly $Cl_{IDB}(EDB) = \{p, q, r, s\}$.

Compute $Cl_{IDB}(EDB) = T'_{IDB}{\uparrow}^{\omega}(EDB)$:

$$\begin{aligned}
T'_{IDB}{\uparrow}^{0}(EDB) &= EDB = \{r, s\}\\
T'_{IDB}{\uparrow}^{1}(EDB) &= T'_{IDB}(\{r, s\}) = \{q\} \cup \{r, s\}\\
T'_{IDB}{\uparrow}^{2}(EDB) &= T'_{IDB}(\{q, r, s\}) = \{p, q\} \cup \{q, r, s\}\\
T'_{IDB}{\uparrow}^{3}(EDB) &= T'_{IDB}(\{p, q, r, s\}) = \{p, q\} \cup \{p, q, r, s\}\\
&\vdots\\
T'_{IDB}{\uparrow}^{\omega}(EDB) &= \{r, s\} \cup \{q, r, s\} \cup \{p, q, r, s\}
\end{aligned}$$

Compare $T_{IDB}{\uparrow}^{\omega}(EDB)$:

$$\begin{aligned}
T_{IDB}{\uparrow}^{0}(EDB) &= EDB = \{r, s\}\\
T_{IDB}{\uparrow}^{1}(EDB) &= T_{IDB}(\{r, s\}) = \{q\}\\
T_{IDB}{\uparrow}^{2}(EDB) &= T_{IDB}(\{q\}) = \emptyset\\
T_{IDB}{\uparrow}^{3}(EDB) &= T_{IDB}(\emptyset) = \emptyset\\
&\vdots\\
T_{IDB}{\uparrow}^{\omega}(EDB) &= \{r, s\} \cup \{q\} = \{q, r, s\}
\end{aligned}$$

(b) Let $IDB = \{\, p \leftarrow q, r\,;\ q \leftarrow s\,;\ r \,\}$ and $EDB = \{\, s \,\}$.

Clearly $Cl_{IDB}(EDB) = \{p, q, r, s\}$.

Compute $Cl_{IDB}(EDB) = T'_{IDB}{\uparrow}^{\omega}(EDB)$:

$$\begin{aligned}
T'_{IDB}{\uparrow}^{0}(EDB) &= EDB = \{s\}\\
T'_{IDB}{\uparrow}^{1}(EDB) &= T'_{IDB}(\{s\}) = \{q, r\} \cup \{s\}\\
T'_{IDB}{\uparrow}^{2}(EDB) &= T'_{IDB}(\{q, r, s\}) = \{p, q, r\} \cup \{q, r, s\}\\
T'_{IDB}{\uparrow}^{3}(EDB) &= T'_{IDB}(\{p, q, r, s\}) = \{p, q, r\} \cup \{p, q, r, s\}\\
&\vdots\\
T'_{IDB}{\uparrow}^{\omega}(EDB) &= \{s\} \cup \{q, r, s\} \cup \{p, q, r, s\}
\end{aligned}$$

Compare $T_{IDB}{\uparrow}^{\omega}(EDB)$:

$$\begin{aligned}
T_{IDB}{\uparrow}^{0}(EDB) &= EDB = \{s\}\\
T_{IDB}{\uparrow}^{1}(EDB) &= T_{IDB}(\{s\}) = \{q, r\}\\
T_{IDB}{\uparrow}^{2}(EDB) &= T_{IDB}(\{q, r\}) = \{p, r\}\\
T_{IDB}{\uparrow}^{3}(EDB) &= T_{IDB}(\{p, r\}) = \{r\}\\
&\vdots\\
T_{IDB}{\uparrow}^{\omega}(EDB) &= \{s\} \cup \{q, r\} \cup \{p, r\} \cup \{r\} = \{p, q, r, s\}
\end{aligned}$$

So here $T_{IDB}{\uparrow}^{\omega}(EDB)$ happens to come out the same as $Cl_{IDB}(EDB)$. Part (a) was intended to show that this is not guaranteed.

Question 2

Part(a): Base case ($n = 0$): $T_P{\uparrow}^{0}(X) = X$ and $T'_P{\uparrow}^{0}(X) = X$.
Inductive hypothesis: suppose $T_P{\uparrow}^{k}(X) \subseteq T'_P{\uparrow}^{k}(X)$.
By monotony of T_P:

$$T_P(T_P{\uparrow}^{k}(X)) \subseteq T_P(T'_P{\uparrow}^{k}(X))$$

Since $T_P(I) \subseteq T_P(I) \cup I$:

$$T_P(T_P{\uparrow}^{k}(X)) \subseteq T'_P(T'_P{\uparrow}^{k}(X))$$

And so:

$$T_P{\uparrow}^{k+1}(X) \subseteq T'_P{\uparrow}^{k+1}(X)$$

That completes the proof.
Further: $T_P{\uparrow}^{\omega}(X) = \bigcup_{n \geq 0} T_P{\uparrow}^{n}(X) \subseteq \bigcup_{n \geq 0} T'_P{\uparrow}^{n}(X) = T'_P{\uparrow}^{\omega}(X)$.

Part (b): to show $T_P{\uparrow}^{n}(\emptyset) \subseteq T_P{\uparrow}^{n+1}(\emptyset)$.
Base case ($n = 0$): $T_P{\uparrow}^{0}(\emptyset) = \emptyset$, and $\emptyset \subseteq T_P{\uparrow}^{1}(\emptyset)$.
Inductive hypothesis: suppose $T_P{\uparrow}^{k}(\emptyset) \subseteq T_P{\uparrow}^{k+1}(\emptyset)$.
By monotony of T_P:

$$\begin{aligned}
T_P(T_P{\uparrow}^{k}(\emptyset)) &\subseteq T_P(T_P{\uparrow}^{k+1}(\emptyset))\\
T_P{\uparrow}^{k+1}(\emptyset) &\subseteq T_P{\uparrow}^{(k+1)+1}(\emptyset)
\end{aligned}$$

That completes the proof.

And hence, to show: $T'_P\!\uparrow^n(\emptyset) = T_P\!\uparrow^n(\emptyset)$.
Base case ($n = 0$): $T'_P\!\uparrow^0(\emptyset) = \emptyset = T_P\!\uparrow^0(\emptyset)$.
Inductive hypothesis: suppose $T'_P\!\uparrow^k(\emptyset) = T_P\!\uparrow^k(\emptyset)$.

$$\begin{aligned}
T'_P\!\uparrow^{k+1}(\emptyset) &= T'_P(T'_P\!\uparrow^k(\emptyset)) \\
&= T'_P(T_P\!\uparrow^k(\emptyset)) \quad \text{(inductive hypothesis)} \\
&= T_P(T_P\!\uparrow^k(\emptyset)) \cup T_P\!\uparrow^k(\emptyset) \\
&= T_P\!\uparrow^{k+1}(\emptyset) \cup T_P\!\uparrow^k(\emptyset) \\
&= T_P\!\uparrow^{k+1}(\emptyset) \quad (\text{because } T_P\!\uparrow^k(\emptyset) \subseteq T_P\!\uparrow^{k+1}(\emptyset))
\end{aligned}$$

That completes the proof.
Further: $T_P\!\uparrow^\omega(\emptyset) = \bigcup_{n\geq 0} T_P\!\uparrow^n(\emptyset) = \bigcup_{n\geq 0} T'_P\!\uparrow^n(\emptyset) = T'_P\!\uparrow^\omega(\emptyset)$.

Question 3 A set X of atoms is a Herbrand model of $IDB \cup EDB$

$$\begin{aligned}
&\text{iff} \quad T_{IDB\cup EDB}(X) \subseteq X && (1)\\
&\text{iff} \quad T_{IDB}(X) \cup EDB \subseteq X && (2)\\
&\text{iff} \quad T_{IDB}(X) \subseteq X \text{ and } EDB \subseteq X && (3)
\end{aligned}$$

So the least set of atoms satisfying (1) is also the least set of atoms satisfying (3), which is $Cl_{IDB}(EDB)$ by definition.
You can also do this by checking that $T'_{IDB\cup EDB}\!\uparrow^n(\emptyset) = T'_{IDB}\!\uparrow^n(EDB)$.

Question 4 Cl_{IDB} is a classical consequence operator:

- $X \subseteq Cl_{IDB}(X)$ by definition of Cl_{IDB}.
- To show $Cl_{IDB}(Cl_{IDB}(X)) \subseteq Cl_{IDB}(X)$:

  (i) $Cl_{IDB}(X) \subseteq Cl_{IDB}(X)$ (obviously)

  (ii) $Cl_{IDB}(X)$ is closed under the rules IDB (by definition).

  But the smallest set of atoms satisfying (i) and (ii) is by definition the closure of $Cl_{IDB}(X)$ under the rules IDB, i.e. $Cl_{IDB}(Cl_{IDB}(X))$. So $Cl_{IDB}(Cl_{IDB}(X)) \subseteq Cl_{IDB}(X)$.
- Suppose $X_1 \subseteq X_2$. Then $X_1 \subseteq X_2 \subseteq Cl_{IDB}(X_2)$. So:

  (i) $X_1 \subseteq Cl_{IDB}(X_2)$

  (ii) $Cl_{IDB}(X_2)$ is closed under the rules IDB (by definition).

  But the smallest set of atoms satisfying (i) and (ii) is by definition $Cl_{IDB}(X_1)$. So $Cl_{IDB}(X_1) \subseteq Cl_{IDB}(X_2)$.

Notice that the third property (monotony of Cl_{IDB}) does not depend on monotony of T_{IDB}. But if IDB are not definite, then T_{IDB} is not necessarily monotonic, and then there is no guarantee that Cl_{IDB} exists.

Question 5 Since $\{r\} \subseteq Cl_{IDB}(\{r\})$ there are four candidates for $Cl_{IDB}(\{r\})$, if it exists:

- $Cl_{IDB}(\{r\}) = \{r\}$?

  No: $\{r\}$ is not closed under rules IDB: $T_{IDB}(\{r\}) = \{p, q\} \not\subseteq \{r\}$.
- $Cl_{IDB}(\{r\}) = \{p, r\}$ or $Cl_{IDB}(\{r\}) = \{q, r\}$?

  No: not unique. $\{p, r\}$ is closed under rules IDB: $T_{IDB}(\{p, r\}) = \{p\} \subseteq \{p, r\}$. Consider the subsets of $\{p, r\}$: $\{r\}$ is not closed under rules IDB (see above), and $\{p\}$ and $\emptyset$ do not contain $\{r\}$.

  However: the same holds for $\{q, r\}$, by the same argument.

  So $\{p, r\}$ is not the *unique* smallest set X such that …, and nor is $\{q, r\}$.
- $Cl_{IDB}(\{r\}) = \{p, q, r\}$?

  No: not minimal. $\{p, q, r\}$ is closed under rules IDB: $T_{IDB}(\{p, q, r\}) = \emptyset \subseteq \{p, q, r\}$. But $\{p, q, r\}$ is not the smallest such set containing $\{r\}$: the subset $\{p, r\}$ is also closed under IDB, for example (above).