491 Knowledge Representation

Tutorial Exercise

Default Logic

(These are not necessarily the kinds of questions you should expect in an exam.)

Question 1 Determine all the extensions of the default theory:

$$D = \{\frac{:\neg q, \neg r}{p}, \frac{:\neg p, \neg r}{q}, \frac{:\neg p, \neg q}{r}\}, \qquad W = \{\}$$

Question 2 Determine all the extensions of the following default theories:

i) $D = \{\frac{p:\neg q}{r}, \frac{p:\neg r}{q}\}, \qquad W = \{p\}$

ii) $D = \{\frac{:q}{p}, \frac{:q}{q}, \frac{:\neg q}{\neg q}, \}, \qquad W = \{\}$

Do this by translating the above into the equivalent logic programs, computing the answer sets, and then checking that the corresponding extensions really are extensions.

Question 3 Construct an example to show $\frac{\alpha : \beta_1, \beta_2}{\gamma}$ and $\frac{\alpha : \beta_1 \wedge \beta_2}{\gamma}$ are not equivalent.

Question 4 Check what is said about extensions in all the examples in the lecture notes, and in particular

- the 'Reggie and Ronnie are not both innocent' example,
- and the example that demonstrates the existence of extensions is not guaranteed, viz.

$$D = \{\frac{:p}{\neg p}\}, \qquad W = \{\}$$

Question 5 (Optional) Prove the following properties of $\mathrm{Cn}_R(W)$ given in the lecture notes:

- $W \subseteq \mathrm{Cn}_R(W)$ ('inclusion')
- $\mathrm{Cn}_R(W) \subseteq \mathrm{Cn}_R(W \cup X)$, any set of formulas X (Cn_R is monotonic)
  which is equivalent to $A \subseteq B \Rightarrow \mathrm{Cn}_R(A) \subseteq \mathrm{Cn}_R(B)$
- $X \subseteq \mathrm{Cn}_R(W) \Rightarrow \mathrm{Cn}_R(W \cup X) \subseteq \mathrm{Cn}_R(W)$ ('cut' *alias* 'cumulative transitivity')
  which is equivalent to $W \subseteq W' \subseteq \mathrm{Cn}_R(W) \Rightarrow \mathrm{Cn}_R(W') \subseteq \mathrm{Cn}_R(W)$
- $\mathrm{Cn}_R(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$ ('closure')
  (Recall that Cn_R monotonic implies 'cut' is equivalent to 'closure')
- Cn_R is a classical consequence relation
- $\mathrm{Th}(W) \subseteq \mathrm{Cn}_R(W)$ ('supraclassical')
- $\mathrm{Cn}_R(W)$ is the smallest set of formulas S such that $S = \mathrm{Th}(W \cup \mathrm{T}_R(S))$.

Question 6 *(from 2009 exam)*
Let D be the following set of default rules, and let $W = \{p \vee q, \neg q\}$.

$$D = \{\frac{p\colon h, s}{h}, \frac{\neg q\colon \neg s, \neg t}{k}, \frac{h\colon \neg s \wedge t}{q \to \neg t}, \frac{h\colon s, t}{p \to t}, \frac{k\colon \neg s, \neg t}{s \to q}\}$$

Show that the following two sets of formulas are extensions of (D, W)

(i) $\mathrm{Th}(\{p \wedge \neg q,\ h \wedge t\})$ (ii) $\mathrm{Th}(\{p \wedge \neg q,\ k \wedge \neg s\})$

and that the following two are not

(iii) $\mathrm{Th}(\{p \wedge \neg q,\ h \wedge \neg s\})$ (iv) $\mathrm{Th}(\{p \wedge \neg q,\ k \wedge t\})$

Show your working.
Without computing the reduct, explain why

(v) $\mathrm{Th}(\{p \wedge \neg q,\ h \wedge t,\ k \wedge \neg s\})$

could not be an extension of (D, W).

Question 7 (Marek & Truszczyński Thm 3.50, p66)
Prove that, if E is an extension of (D, W) then for every $X \subseteq E$, E is an extension of $(D, W \cup X)$.
(This is the essence of cumulative transitivity ('cut') for default logic.)
Hint: You need to show $E = \mathrm{Cn}_{D^E}(W \cup X)$. Show the two halves separately:

$$E \subseteq \mathrm{Cn}_{D^E}(W \cup X) \qquad \text{and} \qquad \mathrm{Cn}_{D^E}(W \cup X) \subseteq E$$

Question 8 (Marek & Truszczyński Ex 3.51, p67)
Let $W = \emptyset$ and

$$D = \{\frac{:\neg b, \neg d}{a}, \frac{:\neg b, \neg d}{c}, \frac{:\neg a, \neg c}{d}, \frac{a:\neg c}{b}\}$$

Confirm that $\mathrm{Th}(\{d\})$ and $\mathrm{Th}(\{a, c\})$ are both extensions of (D, W).
$(D, W \cup \{a\})$ also has two extensions. One is $\mathrm{Th}(\{a, c\})$ (as we know from the previous question). The other one is $\mathrm{Th}(\{a, b\})$. Confirm this. Note that from the previous question, if E is an extension of (D, W) then E is an extension of $(D, W \cup X)$ for every $X \subseteq E$. But (as in the example) there may be other extensions of (D, W) unrelated to E.

Question 9 *Optional:* this is *definitely* not the kind of question you would get in an exam.

In the first set of lecture notes ('Logic Databases (Knowledge Bases)') there were mentioned two possible formalisations of the 'Closed World Assumption':

$$\begin{aligned} cwa(W) &\stackrel{\text{def}}{=} \text{Th}(W \cup \{\neg p \mid p \in atoms(\mathcal{L}) \text{ and } p \notin \text{Th}(W)\}) \\ cwa'(W) &\stackrel{\text{def}}{=} \text{Th}(W \cup \{\neg p \mid p \in atoms(\mathcal{L}) \text{ and } p \notin cwa'(W)\}) \end{aligned}$$

(In the first set of lecture notes I wrote D for 'database'. I have switched to so W as not to confuse with sets of default rules.)

The first definition is Reiter's original formalisation of the 'Closed World Assumption', which pre-dates default logic by about 10 years. It is reasonably straightforward to investigate its formal properties. What about the second definition? Is it even well defined?

The basic idea is familiar enough. We want $cwa'(W)$ to be the *smallest* set of formulas satisfying the equation above; there may be more than one such set $cwa'(W)$, in which case we have to decide whether we want skeptical (intersection) or credulous (union) versions.

As we have seen, 'X is the smallest set of formulas such that [conditions mentioning X]' can be ambiguous. So here is a more careful definition.

Let E be a set of formulas. E is a *cwa*-extension of W when $E = \Gamma(W, E)$ where $\Gamma(W, E)$ is the smallest set S of formulas such that

$$S = \text{Th}(W \cup \{\neg p \mid p \in atoms(\mathcal{L}) \text{ and } p \notin E\})$$

Now the question: $cwa'(W)$ can be formulated as a Reiter default theory? How?

First figure out what the default rules should be and then check that the extensions correspond to the *cwa*-extensions defined above.

(For the last step, look at the last part of Question 5.)

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491 Knowledge Representation

Tutorial Exercise

Default Logic
SOLUTIONS

Question 1 Three extensions: Th($\{p\}$), Th($\{q\}$), Th($\{r\}$).

Question 2

i) The standard translation gives the following (normal) logic program:

$$r \leftarrow p, \texttt{not}\ q$$
$$q \leftarrow p, \texttt{not}\ r$$
$$p$$

Taking the splitting set $\{p\}$ gives (eventually) two stable models (answer sets): $\{p, q\}$ and $\{p, r\}$. So the default theory has two extensions: Th($\{p, q\}$) and Th($\{p, r\}$).

ii) The standard translation gives the following (extended) logic program:

$$p \leftarrow \texttt{not}\ \neg q$$
$$q \leftarrow \texttt{not}\ \neg q$$
$$\neg q \leftarrow \texttt{not}\ q$$

Taking the splitting set $\{q, \neg q\}$ gives two answer sets for the bottom part, $\{q\}$ and $\{\neg q\}$. After simplifying, we get two simplified top parts: $\{p \leftarrow\}$ and $\emptyset$, respectively. So we end up with two answer sets for the complete program: $\{p, q\}$ and $\{\neg q\}$.

The default theory has two extensions: Th($\{p, q\}$) and Th($\{\neg q\}$).

Question 3 Consider $\dfrac{: p, q}{r}$ and $\dfrac{: p \wedge q}{r}$.

The consistency check of the first one requires only that p and q individually are consistent; the second requires that p and q together are consistent. For example, both p and q individually are consistent with $\neg(p \wedge q)$ (which is equivalent to $\neg p \vee \neg q$), whereas $p \wedge q$ is clearly not consistent with $\neg(p \wedge q)$.

So let's take $W = \{\neg p \vee \neg q\}$. Th($\{\neg p \vee \neg q, r\}$) is an extension of W with the first default but is not an extension of W with the second one.

Question 4

- To reduce writing I will rewrite the example in this form:

$$D = \{\frac{s(a) \colon a}{a}, \frac{s(b) \colon b}{b}\}, \qquad W = \{s(a), s(b), \neg(a \wedge b)\}$$

Let's consider $E_1 = \text{Th}(W)$, $E_2 = \text{Th}(W \cup \{a\})$, $E_3 = \text{Th}(W \cup \{a, b\}) = \mathcal{L}$. The case $\text{Th}(W \cup \{b\})$ is obviously symmetric to E_2.

Note that, because $\neg(a \wedge b) \in W$, $E_2 = \text{Th}(W \cup \{a\}) = \text{Th}(W \cup \{a, \neg b\})$.

$D^{E_1} = \{\frac{s(a)}{a}, \frac{s(b)}{b}\}$. $\text{Cn}_{D^{E_1}}(W) = \text{Th}(W \cup \{a, b\}) = \mathcal{L} \neq E_1$.

$D^{E_2} = \{\frac{s(a)}{a}\}$. $\text{Cn}_{D^{E_2}}(W) = \text{Th}(W \cup \{a\}) = E_2$.

$D^{E_3} = \{\}$. $\text{Cn}_{D^{E_3}}(W) = \text{Th}(W) \neq E_3$.

That's all I had in mind, but I suppose that strictly speaking one should check that all possibilities have been exhausted, i.e., that there are no other extensions besides these two. We could try to do this by translating to an extended logic program and finding its answer sets. The formula $\neg(a \wedge b)$ is in the wrong form, but it can be expressed in ASP style in the form $f \leftarrow a, b, \texttt{not}\ f$. (You have seen that trick many times.)

So let's compute the answer sets of:

$$\begin{array}{lll} s(a) & a \leftarrow s(a), \texttt{not}\ \neg a & f \leftarrow a, b, \texttt{not}\ f \\ s(b) & b \leftarrow s(b), \texttt{not}\ \neg b & \end{array}$$

This is easy, using e.g. splitting sets. It's not quite enough though. What this gives us are the extensions of the default theory

$$D' = \{\frac{s(a) \colon a}{a}, \frac{s(b) \colon b}{b}, \frac{a \wedge b \colon \neg f}{f}\}, \qquad W' = \{s(a), s(b)\}$$

I dare say that's equivalent to the default theory (D, W) but it still remains to show they are equivalent. Or: since $\neg(a \wedge b)$ is propositionally equivalent to $\neg a \vee \neg b$, we could also try looking at answer sets of

$$\begin{array}{lll} s(a) & a \leftarrow s(a), \texttt{not}\ \neg a & \neg a \leftarrow \texttt{not}\ \neg b \\ s(b) & b \leftarrow s(b), \texttt{not}\ \neg b & \neg b \leftarrow \texttt{not}\ \neg a \end{array}$$

What's the difference? I wasn't intending you to go this far in this question!

- The existence of extensions is not guaranteed. For example:

$$D = \{\frac{: p}{\neg p}\}, \qquad W = \{\}$$

Suppose E is an extension of (D, W).

  - If $\neg p \in E$ then $D^E = \emptyset$ and $E = \text{Th}(\emptyset)$. But $\neg p \notin \text{Th}(\emptyset)$ so contradiction.
  - If $\neg p \notin E$ then $D^E = \{\frac{}{\neg p}\}$ and $\neg p \in \text{Cn}_{D^E}(\emptyset)$, so contradiction again.

Question 5

- $W \subseteq \text{Cn}_R(W)$ ('inclusion'). Trivial. Immediate from the definition of $\text{Cn}_R(W)$.
- $\text{Cn}_R(W) \subseteq \text{Cn}_R(W \cup X)$, any set of formulas X (Cn_R is monotonic).

$\text{Cn}_R(W) \subseteq \text{Cn}_R(W \cup X)$ because $\text{Cn}_R(W \cup X)$ satisfies the three closure conditions for $\text{Cn}_R(W)$ and by definition $\text{Cn}_R(W)$ is the smallest such set.

By definition, $\text{Cn}_R(W \cup X)$ is closed under Th and closed under T_R. So it just remains to prove $W \subseteq \text{Cn}_R(W \cup X)$. And we get this because $W \subseteq W \cup X \subseteq \text{Cn}_R(W \cup X)$.

- $X \subseteq \mathrm{Cn}_R(W) \;\Rightarrow\; \mathrm{Cn}_R(W \cup X) \subseteq \mathrm{Cn}_R(W)$ ('cut' *alias* 'cumulative transitivity').

  We can get this by showing that if $X \subseteq \mathrm{Cn}_R(W)$ then $\mathrm{Cn}_R(W)$ satisfies the three closure conditions for $\mathrm{Cn}_R(W \cup X)$ because (as above) $\mathrm{Cn}_R(W \cup X)$ is the smallest such set by definition.

  Again, by definition $\mathrm{Cn}_R(W)$ is closed under Th and closed under T_R. So it just remains to show that if $X \subseteq \mathrm{Cn}_R(W)$ then $W \cup X \subseteq \mathrm{Cn}_R(W)$. That's easy: we have $W \subseteq \mathrm{Cn}_R(W)$ and we are assuming $X \subseteq \mathrm{Cn}_R(W)$.

- $\mathrm{Cn}_R(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$ ('closure').

  Recall (first set of lecture notes) that Cn_R monotonic implies 'cut' is equivalent to 'closure'. The proof is given there.

- Cn_R is a classical consequence relation.

  We have shown the three defining properties (inclusion, monotony, and cut or closure/idempotence).

- $\mathrm{Th}(W) \subseteq \mathrm{Cn}_R(W)$ ('supraclassical')

  $W \subseteq \mathrm{Cn}_R(W)$ implies $\mathrm{Th}(W) \subseteq \mathrm{Th}(\mathrm{Cn}_R(W))$ because Th is monotonic. And $\mathrm{Th}(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$ by definition of $\mathrm{Cn}_R(W)$.

- $\mathrm{Cn}_R(W)$ is the smallest set of formulas S such that $S = \mathrm{Th}(W \cup \mathrm{T}_R(S))$.

  This is a bit more involved (longer) but the basic steps are more or less the same.

  We need to show

  1. $\mathrm{Cn}_R(W) = \mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$
  2. If $S = \mathrm{Th}(W \cup \mathrm{T}_R(S))$ then $\mathrm{Cn}_R(W) \subseteq S$.

  First we show 1(a) $\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W))) \subseteq \mathrm{Cn}_R(W)$.
  This is easy: $W \subseteq \mathrm{Cn}_R(W)$ and $\mathrm{T}_R(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$ imply $W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$. Now monotony of Th gives $\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W))) \subseteq \mathrm{Th}(\mathrm{Cn}_R(W))$. But $\mathrm{Th}(\mathrm{Cn}_R(W)) \subseteq \mathrm{Cn}_R(W)$.

  Now we show 1(b) $\mathrm{Cn}_R(W) \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$. We can do this by showing that $\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$ satisfies the closure conditions for $\mathrm{Cn}_R(W)$.

  - $W \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$ (inclusion Th)
  - $\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$ is closed under Th (closure/idempotence Th)
  - $\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$ is closed under the rules R, i.e., $\mathrm{T}_R(\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))) \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$. Because …

    T_R is monotonic, so from part 1(a) we have $\mathrm{T}_R(\mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))) \subseteq \mathrm{T}_R(\mathrm{Cn}_R(W))$.

    But $\mathrm{T}_R(\mathrm{Cn}_R(W)) \subseteq W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)) \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(\mathrm{Cn}_R(W)))$.

  For part (2), we show that if $S = \mathrm{Th}(W \cup \mathrm{T}_R(S))$ then S satisfies the closure conditions for $\mathrm{Cn}_R(W)$ (and hence, by definition, $\mathrm{Cn}_R(W) \subseteq S$):

  - $W \subseteq S$ because $W \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(S))$ (Th inclusion)
  - $\mathrm{Th}(W \cup \mathrm{T}_R(S))$ is closed under Th (Th closure/idempotence)
  - $\mathrm{T}_R(S) \subseteq \mathrm{Th}(W \cup \mathrm{T}_R(S))$ (Th inclusion). But $\mathrm{Th}(W \cup \mathrm{T}_R(S)) = S$, so $\mathrm{T}_R(S) \subseteq S$, as required.

Question 6 *(from 2009 exam)* Let D be as follows, and $W = \{p \vee q, \neg q\}$.

$$D = \{\frac{p\colon h, s}{h}, \quad \frac{\neg q\colon \neg s, \neg t}{k}, \quad \frac{h\colon \neg s \wedge t}{q \to \neg t}, \quad \frac{h\colon s, t}{p \to t}, \quad \frac{k\colon \neg s, \neg t}{s \to q}\}$$

The calculations are routine, and quite straightforward if the definition (reduct + closure) is followed.

(i) Let $E_1 = \mathrm{Th}(\{p \wedge \neg q,\ h \wedge t\})$. Reduct $D^{E_1} = \{\frac{p}{h}, \quad \frac{h}{q \to \neg t}, \quad \frac{h}{p \to t}\}$.

Compute the closure $\mathrm{Cn}_{D^{E_1}}(W)$ using the 'base operator'. Here it is in detail (in more detail than you need to show in an exam). I am writing X_i so as not to confuse with the extensions E_i of parts (i)–(v). Note: $\mathrm{Th}(\{p \vee q, \neg q\}) = \mathrm{Th}(\{p, \neg q\}) = \mathrm{Th}(\{p \wedge \neg q\})$.

$$\begin{aligned}
X_0 &= W = \{p \vee q, \neg q\} \\
X_1 &= \mathrm{B}_{D^{E_1}}(X_0) = \{p \vee q, \neg q\} \cup \{h\} \\
X_2 &= \mathrm{B}_{D^{E_1}}(X_1) = \{p \vee q, \neg q\} \cup \{h\} \cup \{q \to \neg t, p \to t\} \\
X_3 &= \mathrm{B}_{D^{E_1}}(X_2) = X_2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cn}_{D^{E_1}}(\{p \vee q, \neg q\}) &= \mathrm{Th}(\{p \vee q,\ \neg q,\ h,\ q \to \neg t,\ p \to t\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ h,\ t\}) = \mathrm{Th}(\{p \wedge \neg q,\ h \wedge t\}) = E_1
\end{aligned}$$

There is a simple bit of propositional logic in the last step. In case you can't see it:

$$\begin{aligned}
\mathrm{Th}(\{p \vee q,\ \neg q,\ h,\ q \to \neg t,\ p \to t\}) &= \mathrm{Th}(\{p,\ \neg q,\ h,\ q \to \neg t,\ p \to t\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ h,\ t,\ q \to \neg t,\ p \to t\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ h,\ t,\ q \to \neg t\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ h,\ t,\ t \to \neg q\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ h,\ t\})
\end{aligned}$$

(ii) Let $E_2 = \mathrm{Th}(\{p \wedge \neg q,\ k \wedge \neg s\})$. Reduct $D^{E_2} = \{\frac{\neg q}{k}, \quad \frac{h}{q \to \neg t}, \quad \frac{k}{s \to q}\}$.

Compute the closure $\mathrm{Cn}_{D^{E_2}}(W)$ using the 'base operator'.

$$\mathrm{B}_{D^{E_2}}{\uparrow}^{\omega}(W) = \{p \vee q, \neg q\} \cup \{k\} \cup \{s \to q\}$$

$$\begin{aligned}
\mathrm{Cn}_{D^{E_2}}(W) &= \mathrm{Th}(\{p \vee q, \neg q\} \cup \{k\} \cup \{s \to q\}) \\
&= \mathrm{Th}(\{p,\ \neg q,\ k,\ \neg s\}) = E_2
\end{aligned}$$

(iii) Let $E_3 = \mathrm{Th}(\{p \wedge \neg q,\ h \wedge \neg s\})$. Reduct $D^{E_3} = \{\frac{\neg q}{k}, \quad \frac{h}{q \to \neg t}, \quad \frac{k}{s \to q}\}$.

One can already see that $\mathrm{Cn}_{D^{E_3}}(W)$ must contain k but $k \notin E_3$.

(If you want to compute $\mathrm{Cn}_{D^{E_3}}(W)$, notice that $D^{E_3} = D^{E_2}$.)

(iv) Let $E_4 = \mathrm{Th}(\{p \wedge \neg q,\ k \wedge t\})$. Reduct $D^{E_4} = \{\frac{p}{h}, \quad \frac{h}{q \to \neg t}, \quad \frac{h}{p \to t}\}$.

One can see that $\mathrm{Cn}_{D^{E_4}}(W)$ contains h but $h \notin E_4$.

(If you want to compute $\mathrm{Cn}_{D^{E_4}}(W)$, note $D^{E_4} = D^{E_1}$.)

(v) Let $E_5 = \mathrm{Th}(\{p \wedge \neg q,\ h \wedge t,\ k \wedge \neg s\})$.

Expected answer in the exam: $E_1 \subset E_5$ (and $E_2 \subset E_5$). So E_5 could not be an extension of (D, W): E_1 and E_2 are extensions of (D, W) and extensions are *minimal*.

If you want to see it in more detail: $\{p \wedge \neg q,\ h \wedge t\} \subseteq \{p \wedge \neg q,\ h \wedge t,\ k \wedge \neg s\}$. By monotony of Th, $\mathrm{Th}(\{p \wedge \neg q,\ h \wedge t\}) \subseteq \mathrm{Th}(\{p \wedge \neg q,\ h \wedge t,\ k \wedge \neg s\})$. So $E_1 \subseteq E_5$. By a similar argument, $E_2 \subseteq E_5$. But $E_1 \neq E_2$, so $E_1 \subset E_5$ (and $E_2 \subset E_5$). Rest the same.

Question 7 (Marek & Truszczyński Thm 3.50, p66)

Suppose E is an extension of (D, W), i.e., $E = \mathrm{Cn}_{D^E}(W)$. Suppose $X \subseteq E$.

First we want to show $E \subseteq \mathrm{Cn}_{D^E}(W \cup X)$, i.e., $\mathrm{Cn}_{D^E}(W) \subseteq \mathrm{Cn}_{D^E}(W \cup X)$.

$$W \subseteq W \cup X \ \Rightarrow\ \mathrm{Cn}_{D^E}(W) \subseteq \mathrm{Cn}_{D^E}(W \cup X) \qquad (\mathrm{Cn}_{D^E} \text{ monotonic})$$

(Nothing to do with X.)

For the other half:

First, $E = \mathrm{Cn}_{D^E}(W)$ and $X \subseteq E$ so $X \subseteq \mathrm{Cn}_{D^E}(W)$.

And $W \subseteq \mathrm{Cn}_{D^E}(W)$. So $W \cup X \subseteq \mathrm{Cn}_{D^E}(W)$. Then:

$$\begin{aligned} W \cup X \subseteq \mathrm{Cn}_{D^E}(W) &\Rightarrow \mathrm{Cn}_{D^E}(W \cup X) \subseteq \mathrm{Cn}_{D^E}(\mathrm{Cn}_{D^E}(W)) \qquad (\mathrm{Cn}_{D^E} \text{ monotonic}) \\ &\Rightarrow \mathrm{Cn}_{D^E}(W \cup X) \subseteq \mathrm{Cn}_{D^E}(W) \qquad (\mathrm{Cn}_{D^E} \text{ closure/idempotence}) \\ &\Rightarrow \mathrm{Cn}_{D^E}(W \cup X) \subseteq E \end{aligned}$$

Question 8 $D = \{\dfrac{:\neg b, \neg d}{a}, \dfrac{:\neg b, \neg d}{c}, \dfrac{:\neg a, \neg c}{d}, \dfrac{a : \neg c}{b}\}$

$E = \mathrm{Th}(\{d\})$	$E = \mathrm{Th}(\{a, c\})$	$E = \mathrm{Th}(\{a, b\})$
$D^E = \{\frac{}{d}, \frac{a}{b}\}$	$D^E = \{\frac{}{a}, \frac{}{c}\}$	$D^E = \{\frac{a}{b}\}$
$\mathrm{Cn}_{D^E}(\emptyset) = \mathrm{Th}(\{d\})$	$\mathrm{Cn}_{D^E}(\emptyset) = \mathrm{Th}(\{a, c\})$	$\mathrm{Cn}_{D^E}(\emptyset) = \mathrm{Th}(\emptyset)$
$E \in ext(D, \emptyset)$: yes	$E \in ext(D, \emptyset)$: yes	$E \in ext(D, \emptyset)$: no
$\mathrm{Cn}_{D^E}(\{a\}) = \mathrm{Th}(\{a, b, d\})$	$\mathrm{Cn}_{D^E}(\{a\}) = \mathrm{Th}(\{a, c\})$	$\mathrm{Cn}_{D^E}(\{a\}) = \mathrm{Th}(\{a, b\})$
$E \in ext(D, \{a\})$: no	$E \in ext(D, \{a\})$: yes	$E \in ext(D, \{a\})$: yes

Question 9

Consider default rules $D_{\mathrm{cwa}} \stackrel{\mathrm{def}}{=} \{\dfrac{:\neg p}{\neg p} \mid p \in atoms(\mathcal{L})\}$.

Given a set E of formulas we get the following reduct

$$D^E_{\mathrm{cwa}} = \{\frac{}{\neg p} \mid p \in atoms(\mathcal{L}) \text{ and } p \notin E\})$$

So the definition of $\Gamma(W, E)$ can be re-formulated as the smallest set S of formulas such that

$$S \ = \ \mathrm{Th}(W \cup \mathrm{T}_{D^E_{\mathrm{cwa}}}(S))$$

Look at the last part of Question 5. We can see that

$$\Gamma(W, E) \ = \ \mathrm{Cn}_{D^E_{\mathrm{cwa}}}(W)$$

E is defined to be a *cwa*-extension of W when $E = \Gamma(W, E)$, which turns out to be $E = \mathrm{Cn}_{D^E_{\mathrm{cwa}}}(W)$.

In other words, E is a *cwa*-extension of W precisely when E is an extension of the default theory (D_{cwa}, W).