491 Knowledge Representation

Tutorial Exercise

Consequence relations

Notation: When A is a set of sentences (formulas of the language $\mathcal{L}$), $X \vdash A$ is shorthand for $X \vdash \alpha$ for all $\alpha \in A$, i.e., for $A \subseteq \text{Cn}(X)$. $\text{Th}(X)$ is the set of classical truth-functional consequences of X. $X \vdash_{PL} A$ is shorthand for $A \subseteq \text{Th}(X)$.

1. Check that $\text{Th}(X) \stackrel{\text{def}}{=} \{\alpha \in \mathcal{L} \mid X \models \alpha\}$ does indeed satisfy the properties of Th listed in the lecture notes.

   You can ignore compactness. To show closure/idempotence, it is easier to show the more general property 'cut'. (You might want to look at Question 2 first.)

2. *From 2008 Exam:*

   Let *models*(A) denote the set of models of A, i.e., the set of interpretations in which every formula of A is true. One can see that $\alpha \in \text{Th}(A)$ iff $\mathit{models}(A) \subseteq \mathit{models}(\{\alpha\})$, and more generally, that $B \subseteq \text{Th}(A)$ iff $\mathit{models}(A) \subseteq \mathit{models}(B)$. Moreover, if $A \subseteq B$ then $\mathit{models}(B) \subseteq \mathit{models}(A)$ (or if you prefer, $\mathit{models}(A \cup X) \subseteq \mathit{models}(A)$).

   Use these observations to show that Th is a *classical consequence operator*.

   You may find it helpful to prove first the transitivity of Th: $B \subseteq \text{Th}(A)$ implies $\text{Th}(B) \subseteq \text{Th}(A)$, or equivalently, if $B \subseteq \text{Th}(A)$ and $\alpha \in \text{Th}(B)$ then $\alpha \in \text{Th}(A)$.

3. In *preferential entailment*, the consequences of a set A of formulas are defined in terms of some preferred *subset* of models of A: $B \subseteq \text{Cn}_{\text{pref}}(A)$ iff all formulas in B are true in all *preferred* models of A, i.e., iff $\mathit{models}_{\text{pref}}(A) \subseteq \mathit{models}(B)$. The detailed definition of $\mathit{models}_{\text{pref}}$ varies, though it is always the case that $\mathit{models}_{\text{pref}}(A) \subseteq \mathit{models}(A)$.

   Some properties of Cn_{pref} follow immediately. For example:

   - $\mathit{models}_{\text{pref}}(A) \subseteq \mathit{models}(A)$ means that $A \subseteq \text{Cn}_{\text{pref}}(A)$ ('inclusion').

   Also more or less immediate:

   (i) Show that preferential entailment is always 'supraclassical':

   $$\text{Th}(A) \subseteq \text{Cn}_{\text{pref}}(A)$$

   (ii) In general, $A \subseteq B$ does not imply $\mathit{models}_{\text{pref}}(B) \subseteq \mathit{models}_{\text{pref}}(A)$ (or, if you prefer, in general $\mathit{models}_{\text{pref}}(A \cup X) \not\subseteq \mathit{models}_{\text{pref}}(A)$).
   Why does it follow that, in general, preferential entailment is non-monotonic? (Rather obvious.)

   There are some other properties of preferential entailment that can easily be discovered. I will leave them until we discuss properties of non-monotonic consequence relations later in the course.

4. In the lecture notes it is claimed that

   - if $A \vdash B$ and $A \cup B \vdash \alpha$ then $A \vdash \alpha$ ('cut')

   is expressed in terms of Cn as:

   - if $B \subseteq \text{Cn}(A)$ then $\text{Cn}(A \cup B) \subseteq \text{Cn}(A)$

   which in turn is equivalent (assuming $A \subseteq \text{Cn}(A)$) to

   - if $A \subseteq B \subseteq \text{Cn}(A)$ then $\text{Cn}(B) \subseteq \text{Cn}(A)$ ('cumulative transitivity')

   Check this.

5. Suppose Cn is a classical consequence operator. Prove:

   (i) If $A \subseteq D \subseteq B \subseteq \text{Cn}(A)$ then $\text{Cn}(D) = \text{Cn}(B)$
   (ii) If $A \subseteq \text{Cn}(B)$ then $\text{Cn}(A \cup B) = \text{Cn}(B)$
   (iii) If $X \vdash \alpha$ and $X \cup \{\alpha\} \vdash \beta$ then $X \vdash \beta$
   (iv) $\text{Cn}(B) = \text{Cn}(D)$ iff $B \subseteq \text{Cn}(D)$ and $D \subseteq \text{Cn}(B)$ (i.e., $D \vdash B$ and $B \vdash D$)
   (v) $\vdash$ is transitive: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$

   (These properties are listed here just for the exercise. You don't have to memorise them or anything like that. But still, read what they are saying and see if it makes sense. You could draw a picture for (i). (ii) and (iii) you have seen already on this sheet.)

6. Suppose Cn is also *supraclassical*, i.e. $\text{Th}(A) \subseteq \text{Cn}(A)$ for all A. Prove:

   (i) If $X \vdash_{PL} Y$ then $A \cup Y \vdash \alpha$ implies $A \cup X \vdash \alpha$
   (or equivalently, $\text{Cn}(A \cup \text{Th}(X)) \subseteq \text{Cn}(A \cup X)$)
   (ii) Hence, $\text{Cn}(A \cup \{\alpha, \beta\}) = \text{Cn}(A \cup \{\alpha \wedge \beta\})$
   (and so also more generally: $\text{Cn}(\{\alpha_1, \alpha_2, \ldots \alpha_n\}) = \text{Cn}(\{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n\})$)
   (iii) If $X \vdash_{PL} Y$ then $A \vdash X$ implies $A \vdash Y$
   (iv) Hence, for finite A, $A \subseteq \text{Cn}(D)$ iff $D \vdash \bigwedge A$

7. Suppose T is a set of formulas of the language $\mathcal{L}$. Consider the operator Cn_T defined as follows:

   $$\text{Cn}_T(A) \stackrel{\text{def}}{=} \text{Th}(T \cup A)$$

   (You can think of formulas T as expressing some fixed 'background' knowledge.)

   Show that Cn_T is a classical consequence operator and that it satisfies the further properties of supraclassicality, deduction, and compactness.

   Hence show that

   $$X \cup \{\alpha\} \vdash_T \beta \text{ and } X \cup \{\gamma\} \vdash_T \beta \quad \text{iff} \quad X \cup \{\alpha \vee \gamma\} \vdash_T \beta$$