491 Knowledge Representation

Stratified logic programs

Marek Sergot
Department of Computing
Imperial College, London

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Several groups of people — Apt, Blair, and Walker, Przymusiński, van Gelder, Naqui, Topor (and others) — came up with the same idea at about the same time. Some of them were working in logic programming, some of them were working in databases. Either way, the same basic idea:

- restrict attention to a special class of *stratified* logic programs/databases;
- then show that every stratified logic program/database has a unique model which is minimal and supported, and moreover which can be obtained by modifying the usual fixpoint construction $T_P{\uparrow}^\omega$ in a very natural way.

I follow Apt, Blair and Walker's (ABW) presentation here. I omit most proofs and concentrate on motivating the basic ideas and stating the key results.

> Apt, K.R., Blair, H.A. and Walker, A. Towards a Theory of Declarative Knowledge.
> In *Foundations of Deductive Databases and Logic Programming*, Minker, J., ed.
> Morgan Kaufmann Publishers Inc., 1988, pp 89–148.

The notion of stratified program/database is very important. Very similar ideas have appeared elsewhere ('separability' in the literature on circumscription, for example).

Minimal supported (Herbrand) models

As already observed, a *normal* logic program/database such as P:

$p(1) \leftarrow$
$q(2) \leftarrow$
$r(X) \leftarrow \text{not } q(X)$

has two *different* minimal Herbrand models:

$$\{p(1), q(2), r(1)\} \quad \text{and} \quad \{p(1), q(2), q(1)\}$$

The first of these is supported and the second one is not. (The atom $q(1)$ in the second model has no support.)

A natural candidate for the semantics of a normal logic program/database is the

supported minimal (Herbrand) model.

We just need to figure out when such a model exists, and when it is unique.

We know that, even for a normal logic program P:

- $T_P(I) \subseteq I$ means that I is a *model* of P (I is closed under T_P);
- $T_P(I) \supseteq I$ means that I is *supported*.

Unfortunately, we cannot find a fixpoint $T_P(I) = I$ just by constructing $T_P{\uparrow}^\omega$ (or $T'_P{\uparrow}^\omega(\emptyset)$) because T_P is not sufficiently well behaved when P contains clauses with occurrences of negation-by-failure not in the body. T_P is not even monotonic in that case. But imposing a reasonable restriction on the form of P allows us to devise a natural means of constructing fixpoints of T_P.

Stratified programs/databases: outline

Suppose we do the following:

P_0 – any set of *definite* clauses (no negation), whose unique minimal model is $\mathbf{M}(P_0) = T_{P_0}{\uparrow}^\omega$ (as usual – no problem). Call this model M_0.

P_1 – any set of clauses (including negative literals in bodies) that define new predicates, *except that* negative literals are only allowed to refer to predicates that are *already defined* in P_0. Let M_1 be the closure of facts M_0 under the rules P_1: since all negative conditions in clauses of P_1 are fully determined by M_0, we can expect that this closure will be $M_1 = T'_{P_1}{\uparrow}^\omega(M_0)$.

P_2 – any set of clauses that define new predicates, *except that* negative literals are only allowed to refer to predicates that are *already defined* in $P_0 \cup P_1$. Let M_2 be the closure of facts M_1 under the rules P_2: we can expect that this closure will be $M_2 = T'_{P_2}{\uparrow}^\omega(M_1)$.

⋮

etc

We would end up with a normal logic program/database P that is made up of several distinct strata, like this:

$$P = P_0 \cup P_1 \cup \cdots \cup P_n \qquad (P_i \text{ and } P_j \text{ disjoint for all } i \neq j)$$

Such a program/database P is *stratified.* If our intuition is correct, then the last $M_n = T'_{P_n}{\uparrow}^\omega(M_{n-1})$ will be a model of P. This remains to be checked. (M_n will turn out to be a minimal supported model of P.)

The above description suggests we construct a stratified program/database P by successively constructing a sequence of strata. But this is not necessary. A stratified program/database P is one that *could* have been constructed in this way. It is only necessary that it can be partitioned into strata as described above, not that it was actually constructed in this way.

Stratified programs/databases

There are several alternative (and equivalent) definitions of *stratified*. Here below is one. Note the key feature: a stratified program forbids recursion 'inside negation'. For example, any program containing a clause of the form

$$p \leftarrow q, \text{not } p$$

is not stratified. Nor is any program containing clauses of the form

$$\begin{array}{l} p \leftarrow q, \text{not } r \\ r \leftarrow s, p \end{array}$$

And so on.

Preliminary definitions

- If P contains a clause of the form

  $$A \leftarrow \ldots, p(\ldots), \ldots$$

  then predicate p *occurs positively* in the clause.
- If P contains a clause of the form

  $$A \leftarrow \ldots, \text{not } p(\ldots), \ldots$$

  then predicate p *occurs negatively* in the clause.

Note that a predicate could occur both positively and negatively, even in the same clause.

Definition A normal logic program/database P is *stratified* when there is a partition

$$P = P_0 \cup P_1 \cup \cdots \cup P_n \qquad (P_i \text{ and } P_j \text{ disjoint for all } i \neq j)$$

such that, for every predicate p

- the definition of p (all clauses with p in the head) is contained in one of the partitions/strata P_i

and, for each $1 \leq i \leq n$:

- if a predicate occurs *positively* in a clause of P_i then its definition is contained within

  $$\textstyle\bigcup_{j \leq i} P_j$$
- if a predicate occurs *negatively* in a clause of P_i then its definition is contained within

  $$\textstyle\bigcup_{j < i} P_j$$

Note that a program/database is stratified if there is *any* such partition.

Notice also that the above definition does allow normal logic programs in the bottom stratum, as long as the negated predicates are undefined.

Example Logic program/database P:

$$\begin{array}{l} p(X) \leftarrow q(X), \text{not } r(X) \\ p(X) \leftarrow q(X), \text{not } t(X) \\ r(X) \leftarrow s(X), \text{not } t(X) \\ t(a) \leftarrow \\ s(a) \leftarrow \\ s(b) \leftarrow \\ q(a) \leftarrow \end{array}$$

Here is one stratification of P:

$$\begin{aligned} P = &\{p(X) \leftarrow q(X), \text{not } r(X),\ p(X) \leftarrow q(X), \text{not } t(X)\} \cup \\ &\{r(X) \leftarrow s(X), \text{not } t(X)\} \cup \\ &\{t(a) \leftarrow,\ s(a) \leftarrow,\ s(b) \leftarrow,\ q(a) \leftarrow\} \end{aligned}$$

Here is another stratification of P:

$$\begin{aligned} P = &\{p(X) \leftarrow q(X), \text{not } r(X),\ p(X) \leftarrow q(X), \text{not } t(X)\} \cup \\ &\{r(X) \leftarrow s(X), \text{not } t(X)\} \cup \\ &\{t(a) \leftarrow\} \cup \\ &\{s(a) \leftarrow,\ s(b) \leftarrow,\ q(a) \leftarrow\} \end{aligned}$$

Here is another:

$$\begin{aligned} P = &\{p(X) \leftarrow q(X), \text{not } r(X),\ p(X) \leftarrow q(X), \text{not } t(X)\} \cup \\ &\{r(X) \leftarrow s(X), \text{not } t(X),\ s(a) \leftarrow,\ s(b) \leftarrow,\ q(a) \leftarrow\} \cup \\ &\{t(a) \leftarrow\} \end{aligned}$$

etc, etc.

Example Logic program/database P:

$$\begin{array}{l} p \leftarrow \text{not } q, \text{not } r \\ q \leftarrow \text{not } s \\ r \leftarrow \text{not } t \\ s \leftarrow \\ t \leftarrow \end{array}$$

Here is one stratification of P:

$$\begin{aligned} P = &\{p \leftarrow \text{not } q, \text{not } r\} \cup \\ &\{q \leftarrow \text{not } s\} \cup \\ &\{r \leftarrow \text{not } t\} \cup \\ &\{s \leftarrow,\ t \leftarrow\} \end{aligned}$$

Here is another stratification of P:

$$\begin{aligned} P =& \{p \leftarrow \text{not } q, \text{not } r\} \cup \\ & \{q \leftarrow \text{not } s,\ r \leftarrow \text{not } t\} \cup \\ & \{s \leftarrow,\ t \leftarrow\} \end{aligned}$$

etc, etc.

Example Logic program/database P:

$p \leftarrow q, \text{not } r$
$r \leftarrow s, \text{not } p$
$q \leftarrow$
$s \leftarrow$

This program cannot be stratified. Notice that it contains a 'recursion through negation'.

Example Logic program/database P:

$p \leftarrow q$
$q \leftarrow \text{not } r$

P is stratified. One stratum is enough:

$$P = \{p \leftarrow q,\ q \leftarrow \text{not } r\}$$

It is easy to check that the stratification criteria are satisfied, because r is undefined in P.

If you prefer, you could think of this as if there were a bottom stratum $\emptyset$, and r is 'defined' in $\emptyset$.

Dependency graphs

Here is a way of visualising the stratification conditions which some people find helpful.
Apt, K.R., Blair, H.A. and Walker, A. Towards a Theory of Declarative Knowledge. In *Foundations of Deductive Databases and Logic Programming*, Minker, J., ed. Morgan Kaufmann, 1988, pp 89–148.

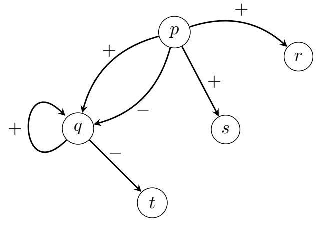
- If program P contains a clause of the form
$$p(\ldots) \leftarrow \ldots, q(\ldots), \ldots$$
then predicate p *refers (+) to* q (or '*depends (+) on* q' if you prefer).
- If P contains a clause of the form
$$p(\ldots) \leftarrow \ldots, \text{not } q(\ldots), \ldots$$
then predicate p *refers (-) to* q (or '*depends (-) on* q').

Now the *dependency graph* for P:

- predicates at nodes;
- *refers to* edges labelled $(+/-)$ between nodes.

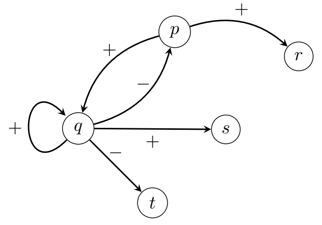
Example

$p \leftarrow q,\ r$
$p \leftarrow \text{not } q,\ s$
$q \leftarrow q,\ \text{not } t$



Example

$p \leftarrow q,\ r$
$q \leftarrow \text{not } p,\ s$
$q \leftarrow q,\ \text{not } t$



(Thanks to Silvia Dobrota (MEng4, 2013-14) for correcting a mistake in the original graph.)

Lemma The logic program P is stratified iff the dependency graph for P contains no cycles containing a negative edge.

Fixpoint semantics for stratified programs

T_P is the immediate consequence operator for the normal logic program/database P. We use the 'progressive' version:

$$T'_P(I) \overset{\text{def}}{=} T_P(I) \cup I$$

Here is the recipe:

- Stratify the program

$$P = P_0 \cup P_1 \cup \cdots \cup P_n \qquad (P_i \text{ and } P_j \text{ disjoint for all } i \neq j)$$

- Now construct

$$\begin{aligned} M_0 &= T'_{P_0}{\uparrow}^{\omega}(\emptyset) && \textit{only the clauses in } P_0 \textit{ !!} \\ M_1 &= T'_{P_1}{\uparrow}^{\omega}(M_0) && \textit{only the clauses in } P_1 \textit{ !!} \\ &\vdots \\ M_n &= T'_{P_n}{\uparrow}^{\omega}(M_{n-1}) \end{aligned}$$

- Take $M_P = M_n$ as the intended semantics of P.

I often refer to this as the 'ABW iterated fixpoint construction'.

It still remains to establish that M_P is a model of P. This and other key results follow after some examples.

Example Logic program/database P:

$$\begin{array}{l} p(X) \leftarrow q(X), \text{not } r(X) \\ r(X) \leftarrow s(X), \text{not } t(X) \\ t(a) \leftarrow \\ s(a) \leftarrow \\ s(b) \leftarrow \\ q(a) \leftarrow \end{array}$$

One possible stratification of P:

$$\begin{aligned} P = &\{p(X) \leftarrow q(X), \text{not } r(X)\} \ \cup && \text{“}P_2\text{”} \\ &\{r(X) \leftarrow s(X), \text{not } t(X)\} \ \cup && \text{“}P_1\text{”} \\ &\{t(a) \leftarrow,\ s(a) \leftarrow,\ s(b) \leftarrow,\ q(a) \leftarrow\} && \text{“}P_0\text{”} \end{aligned}$$

Applying the recipe:

$$\begin{aligned} M_0 &= T'_{P_0}{\uparrow}^{\omega}(\emptyset) = \{t(a),\ s(a),\ s(b),\ q(a)\} \\ M_1 &= T'_{P_1}{\uparrow}^{\omega}(M_0) = \{r(b),\ t(a),\ s(a),\ s(b),\ q(a)\} \\ M_2 &= T'_{P_2}{\uparrow}^{\omega}(M_1) = \{p(a),\ r(b),\ t(a),\ s(a),\ s(b),\ q(a)\} \end{aligned}$$

M_2 is a fixpoint of T_P. (Check it!) So M_2 is a supported model of P.

Suppose we take some other stratification of P? (Some were given earlier for this example.) If you apply the recipe, you will see that you get the same model, whatever stratification of P you take. This is not a coincidence.

Example Here is another of the earlier examples. Logic program/database P:

$$\begin{array}{l} p \leftarrow q \\ q \leftarrow \text{not } r \end{array}$$

To stratify P, one stratum is enough (because r is not defined in P).
Applying the recipe:

$$\begin{aligned} T'_P{\uparrow}^0(\emptyset) &= \emptyset \\ T'_P{\uparrow}^1(\emptyset) &= \{q\} \\ T'_P{\uparrow}^2(\emptyset) &= \{p, q\} \\ &\vdots \\ T'_P{\uparrow}^{\omega}(\emptyset) &= \{p, q\} \end{aligned}$$

$\{p, q\}$ is a fixpoint of T_P (check) and so a supported model of P.

Stratified programs: Key results

Theorem For a stratified normal logic program/database$P$, the 'ABW iterated fixpoint' construction described above constructs a set of ground atoms M_P such that:

- M_P is a fixpoint of T_P
- M_P is a *model* of P
- M_P is *supported*
- M_P is a *minimal* model of P

Theorem M_P is *independent* of the way the program P is stratified.

That's good! And finally there is a link to the Clark completion.

Theorem M_P is a model of the Clark completion comp(P).

Implications for SLDNF

Implication 1
M_P is only *one* model of comp(P). There may be other models. So: in general, more answers to a query will be correct with respect to the ABW semantics M_P (true in the model M_P) than are correct with respect to comp(P) (where we require true in *all* models of comp(P) and not just M_P).
We know that SLDNF is incomplete with respect to the Clark completion comp(P) – there may be consequences of comp(P) that cannot be computed using the SLDNF procedure. So SLDNF is 'even more incomplete' with respect to the ABW semantics M_P of P.

Implication 2
Since M_P is a model of comp(P), for every stratified program P

- comp(P) is *consistent*.

Locally stratified programs

Consider this program

$$p(a) \leftarrow \text{not } p(c)$$
$$p(b) \leftarrow \text{not } p(c)$$

It is clearly *not stratified*. But suppose it had been written with three different 0-ary predicates instead of one unary predicate, like this:

$$p_a \leftarrow \text{not } p_c$$
$$p_b \leftarrow \text{not } p_c$$

This program *is* stratified.

This suggests that we could have a slightly more general notion of stratification – *locally stratified* (Przymusiński). Locally stratified does to ground atoms and ground instances of clauses what stratified does to predicates and clauses with variables.

Locally stratified program/database P:

- Replace P by all ground instances of its clauses.
- Instead of assigning every *predicate* to a partition/stratum of P, assign every *ground atom* to a partition/stratum of P: the definition of that atom (the ground clauses with that atom as head) are put in this partition/stratum.
- Re-express the stratification criteria in terms of ground atoms rather than predicates: if a ground atom occurs positively in a clause in partition/stratum P_i, the definition of that ground atom must be contained in $\bigcup_{j \le i} P_j$; and if a ground atom occurs negatively in a clause in partition/stratum P_i, the definition of that ground atom must be contained in $\bigcup_{j<i} P_j$.

Theorem (Obvious)
Every stratified program is locally stratified.

Remark The generalisation to locally stratified is not particularly useful in practice, but there are references to it in the literature, and it can be a very useful device when trying to prove, e.g., the existence of a supported minimal model of a logic program/database.

‘Preferential entailment’

Here is an idea that comes up frequently in the study of non-monotonic and default reasoning.

Let A be any set of formulas (not necessarily clauses) of some language and α any formula of that language.

Standard logical consequence:

- $A \models \alpha$ — α is true in all models of A

‘Preferential entailment’:

- $A \models_{\text{pref}} \alpha$ — α is true in all special *preferred* models of A

Since the special ‘preferred’ models of A are a subset (usually a proper subset) of the set of all models of A, more formulas will be ‘preferentially entailed’ by A than are logical consequences of A.

The exact specification of special/preferred differs depending on context and application.

Here are some possibilities straightaway. Let P be a normal logic program/database. The special, preferred models of P could be, e.g.

- the models of comp(P)
- the minimal models of P
- the minimal supported models of P
- the ABW iterated fixpoint model, if it exists

(There are other possibilities)

Example

Compare some alternative semantics for the following databases/programs, in each case considering what answers are correct for queries ?-p and ?-q.

$DB_1 = \{q \leftarrow \text{not } p\}$

i) logical consequences of the Clark completion comp(DB_1)

   comp$(DB_1) = \{q \leftrightarrow \neg p,\ \neg p\}$.

   comp$(DB_1) \models \neg p$, comp$(DB_1) \models q$.

   So ?-q gets ‘yes’ and ?-p gets ‘no’.

ii) minimal Herbrand models

   $\{p\}$ and $\{q\}$ are the minimal models. p is true in one and false in the other. So p is not ‘preferentially entailed’ by the minimal model semantics, and neither is $\neg p$. Likewise for q and $\neg q$.

   Queries ?-p and ?-q both get ‘don’t know’.

iii) minimal supported Herbrand models

   Of the two minimal models, only $\{q\}$ is supported. So according to this semantics: ?-q gets ‘yes’ and ?-p gets ‘no’.

iv) ABW iterated fixpoint model

   $T'_{DB_1}\uparrow^{\omega}(\emptyset) = \{q\}$.

   So ?-q gets ‘yes’ and ?-p gets ‘no’.

$DB_2 = \{q \leftarrow \text{not } p,\ p \leftarrow p\}$

i) logical consequences of the Clark completion comp(DB_2)

   comp$(DB_2) = \{q \leftrightarrow \neg p,\ p \leftrightarrow p\}$.

   Neither p nor $\neg p$ are consequences of comp(DB_2). Likewise for q and $\neg q$.

   So queries ?-p and ?-q both get ‘don’t know’.

ii) minimal Herbrand models

   $\{p\}$ and $\{q\}$ are the minimal models. So just as for DB_1.

   ?-p and ?-q both get ‘don’t know’.

iii) minimal supported Herbrand models

   Unlike DB_1, now both minimal models $\{p\}$ and $\{q\}$ are supported. So:

   ?-p and ?-q both get ‘don’t know’.

iv) ABW iterated fixpoint model

DB_2 can be stratified: $P_0 = \{p \leftarrow p\}$, $P_1 = \{q \leftarrow \text{not } p\}$.

$T'_{P_1}\!\uparrow^{\omega}(T'_{P_0}\!\uparrow^{\omega}(\emptyset)) = T'_{P_1}\!\uparrow^{\omega}(\emptyset) = \{q\}$. Same as DB_1.

$DB_3 = \{q \leftarrow \text{not } p,\ p \leftarrow \text{not } q\}$

i) logical consequences of the Clark completion comp(DB_3)

comp(DB_3) $= \{q \leftrightarrow \neg p,\ p \leftrightarrow \neg q\}$.

Neither p nor $\neg p$ are consequences of comp(DB_3). Likewise for q and $\neg q$.

So queries ?-p and ?-q both get 'don't know'.

ii) minimal Herbrand models

$\{p\}$ and $\{q\}$ are the minimal models. Same as DB_2.

Queries ?-p and ?-q both get 'don't know'.

iii) minimal supported Herbrand models

$\{p\}$ and $\{q\}$ are the minimal models. Both are supported. Same as DB_2.

Queries ?-p and ?-q both get 'don't know'.

iii) ABW iterated fixpoint model

DB_3 is not stratified. The ABW semantics is not defined.

Final remark

Theorem For any finite normal logic program P, a set of atoms is a model of comp(P) iff it is a supported model of P.

The proof is easy:
Proof comp(P) consists of the following, for every atom α:

$$\begin{aligned}
&\alpha \leftarrow \mathit{Body}_1\\
&\vdots\\
&\alpha \leftarrow \mathit{Body}_k\\
&\alpha \rightarrow \mathit{Body}_1 \vee \cdots \vee \mathit{Body}_k
\end{aligned}$$

where $\alpha \leftarrow \mathit{Body}_i$ $(1 \le i \le k)$ are all the clauses in P with atom α in the head, or

$$\neg\alpha$$

if there are no clauses in P with atom α in the head.

So, a set X of atoms satisfies (is a model of) comp(P) iff for every atom α:

(1) for every clause $\alpha \leftarrow \mathit{Body}_i$ in P, if $X \models \mathit{Body}_i$ then $\alpha \in X$;

(2) if $\alpha \in X$ then $X \models \mathit{Body}_i$ for some clause $\alpha \leftarrow \mathit{Body}_i$ in P.

(1) says that X is a model of P (X is closed under the clauses of P); (2) says that X is supported by P.