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A hierarchy of reverse bisimulations on stable configuration structures

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Van Glabbeek and Goltz (and later Fecher) have investigated the relationships between various equivalences on stable configuration structures, including interleaving bisimulation (IB), step bisimulation (SB), pomset bisimulation and hereditary history-preserving (H-H) bisimulation. Since H-H bisimulation may be characterised by the use of reverse as well as forward transitions, it is of interest to investigate these and other forms of bisimulations where both forward and reverse transitions are allowed. Bednarczyk asked whether SB with reverse steps is as strong as H-H bisimulation. We answer this question negatively. We give various characterisations of SB with reverse steps, showing that forward steps do not add power. We strengthen Bednarczyk's result that, in the absence of auto-concurrency, reverse IB is as strong as H-H bisimulation, by showing that we need only exclude auto-concurrent events at the same depth in the configuration.

We consider several other forms of observations of reversible behaviour and define a wide range of bisimulations by mixing the forward and reverse observations. We investigate the power of these bisimulations and represent the relationships between them as a hierarchy with IB at the bottom and H-H at the top.

1. Introduction

Van Glabbeek and Goltz (2001), and later Fecher (2004), investigated the relationships between various equivalences on configuration structures, including interleaving bisimulation (IB), step bisimulation (SB), pomset bisimulation (PB) and hereditary history-preserving (H-H) bisimulation. Since H-H bisimulation may be characterised by the use of reverse as well as forward transitions, it is of interest to investigate forms of IB, SB and PB where both forward and reverse transitions are allowed. We shall look at the various possible combinations, such as forward pomsets with reverse steps, and map out the hierarchy of equivalences induced by such notions.

In this paper we follow van Glabbeek and Goltz in adopting as our model of concurrency the framework of *stable configuration structures*. These can be regarded

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as a more abstract version of stable event structures (Winskel 1987). A *configuration* is a set of events that form a possible run within an event structure. Using stable configuration structures means that our results can be directly related to those of van Glabbeek and Goltz, and Fecher. In our earlier work on reversible computation (Phillips and Ulidowski 2007a), we employed *prime* event structures, which are a special case of stable event structures. Prime event structures are technically simpler; they have a global causal ordering on events, whereas the causal orderings for stable event structures are parametrised by the configuration. Stable event structures have the advantage over prime event structures that such operations as sequential, and parallel composition can be modelled more easily. See, for example, van Glabbeek and Goltz (2001) for further discussion of the merits of the various forms of event structure.

The question then arises as to when two configuration structures should be regarded as equivalent. However, we do not want to distinguish configuration structures merely on the basis that they have different sets of events. So we assume that events come with a labelling (we use a, b, c for labels), and that events with the same label are equivalent. The idea is that the labels represent what we can *observe* of the events. Simple configuration structures can be written in a CCS-style notation (Milner 1989), with conflict represented by choice ($+$), concurrency by parallel composition ($|$) and causal ordering by action prefixing (.). As a simple example, the law $a + a = a$ will hold for any reasonable equivalence on configuration structures, even though the configuration structure represented by $a+a$ has two (conflicting) events while a only has one.

Hereditary history-preserving (H-H) bisimulation is considered to be the finest well-behaved true concurrency equivalence relation that 'completely respects the causal and branching structure of concurrent systems and their interplay' (van Glabbeek and Goltz 2001). Here 'well-behaved' means that it preserves refinement of actions. H-H bisimulation was first proposed in Bednarczyk (1991). It can be regarded as the canonical equivalence on event structures in view of its category-theoretical characterisation as open map bisimulation with labelled partial orders as the observations (Joyal *et al.* 1996). H-H bisimulation is defined over configuration structures and, in addition to matching configurations and the transitions between configurations, it also keeps a history of the matched events along matching computations. This is achieved by means of label-preserving and order-preserving isomorphisms between the elapsed events of the two configuration structures. H-H bisimulation and its decidability were further researched by Fröschle in Fröschle (2004) and her subsequent papers, for example, Fröschle and Lasota (2005), and in Jurdzinski *et al.* (2003).

The work reported in the current paper was all motivated by the issue of what observations we can make of concurrent processes. The most fundamental observation is of the label of a single event. This yields interleaving bisimulation (IB), which, as its name indicates, is unable to make true concurrency distinctions, such as between $a|b$ and $a.b + b.a$. A popular method for increasing the discriminating power of a bisimulation is to generalise single action transitions $X \xrightarrow{a} X'$ to transitions $X \xrightarrow{\mu} X'$ where μ is a structure richer than a single action (Pomello 1986; Boudol and Castellani 1987; Cherief 1992). It could be a set of events occurring concurrently (or a multiset of action labels of these events), a so-called 'step', giving step bisimulation (SB), or even a pomset, giving pomset bisimulation (PB). Here steps and pomsets are more refined observations that we can make of a process.

As we have just stated, H-H can be regarded as the canonical true concurrency equivalence. However, we do not regard this as being observational in character, due to its reliance on order isomorphisms. So the question then arises (and this is the main question underlying the work in this paper) as to whether there is an observational characterisation of H-H. This is somewhat reminiscent of the programme carried out by Abramsky to see whether observational equivalence (for, say, CCS) can be characterised as a testing equivalence if the tests are sufficiently powerful (Abramsky 1987).

H-H bisimulation requires the isomorphisms to be consistent under both forward and backward transitions between configurations. It is therefore natural to look at the power of forms of IB, SB and PB where reverse transitions are allowed in addition to forward ones. Adding reverse transitions to IB gives us what we shall call *reverse interleaving IB* (RI-IB). Adding reverse step transitions to SB gives us *reverse step SB* (RS-SB). Adding reverse pomset transitions to PB gives us *reverse pomset PB* (RP-PB). These three equivalences are the main ones studied in this paper, reflecting the possibility of making observations in both forward and reverse directions. We wish to investigate what extra power this gives.

RI-IB has already been investigated in Bednarczyk (1991). When there is no auto-concurrency (concurrent events with the same label), RI-IB equivalence is finer than many true concurrency bisimulations (van Glabbeek and Goltz 2001) up to and including history-preserving bisimulation (De Nicola *et al.* 1987; Rabinovich and Trakhtenbrot 1988). The so-called *absorption law* (Boudol and Castellani 1987; Bednarczyk 1991; van Glabbeek and Goltz 2001)

$$(a \mid (b+c)) + (a \mid b) + ((a+c) \mid b) = (a \mid (b+c)) + ((a+c) \mid b)$$

is not valid for RI-IB equivalence: If one performs a and then b with the $a \mid b$ component on the left, these must be matched by the a and then the b of the $((a+c) \mid b)$ summand on the right (matching it with the a of $(a \mid (b+c))$ is wrong since after this a is performed, no c is possible after a in $a \mid b$). The right-hand side can now reverse a and do a c (still using the same summand as all other summands are disabled), but the left-hand side cannot match this.

In fact, Bednarczyk proved that in the absence of auto-concurrency, RI-IB equivalence has the same power as H-H equivalence (on prime event structures). We shall prove (Theorem 9.7) an extension of this result: RI-IB equivalence has the same power as H-H equivalence in the absence of *equidepth* auto-concurrency, that is, when we cannot have two events with the same label occurring at the same depth within a configuration. The *depth* of an event e is the length of the longest causal chain of events up to and including e.

When auto-concurrency is present, RI-IB is unable to distinguish between such simple processes as $a \mid a$ and $a.a$. This motivates the study of RS-SB, which was briefly mentioned by Bednarczyk, who asked whether RS-SB is as fine as H-H bisimulation. We answer this question in the negative. In fact, we can go further and show that RP-PB is strictly finer than RS-SB, and H-H is strictly finer than RP-PB. Of course, this means that adding reverse observations (or at least those considered here) is not sufficient to give an observational characterisation of H-H. Nevertheless, it is clear that reverse observations do add considerable power, as shown by the Absorption Law example above. See Figure 25 in Section 10 for a diagram showing how RI-IB, RS-SB and RP-PB relate to the hierarchy of forward-only equivalences established by van Glabbeek and Goltz and by Fecher.

If we allow forward step transitions, but only single reverse transitions, we can already distinguish $a \mid a$ from $a.a$ very easily: $a \mid a$ can do an $\{a, a\}$ step whereas $a.a$ cannot. However, we cannot distinguish $a|a$ from $(a|a) + a.a$. Here the reverse steps are needed, and $a|a$ is not equivalent to $(a|a) + a.a$ for RS-IB; we can perform two as in sequence in $(a|a) + a.a$ and get to a configuration where we cannot do a reverse step $\{a, a\}$, unlike the case for $a|a$.

We show that all the power of RS-SB equivalence resides in the reverse step transitions, with the forward steps being dispensable (though, of course, often useful in examples). In fact, the reverse steps can be restricted to those that are *homogeneous*, by which we mean that all events have the same label (thus the events are auto-concurrent) (Theorem 5.10). One can even restrict attention to reverse homogeneous *equidepth* steps, where all events have the same depth. We also show that RS-SB equivalence preserves depth, in the sense that corresponding events must have the same depth.

We introduce the notion of *depth-respecting* bisimulation (DB). This is inspired by the idea that instead of observing steps or pomsets, one can give oneself the power to observe the depth of events. In the forward-only direction, it turns out that DB is strictly stronger than SB. However, we prove that *reverse* depth-respecting DB (RD-DB) has the same power as RS-SB (Theorem 5.24). We also use DB to give a simpler proof of Fecher's result that weak history-preserving equivalence is included in SB equivalence. We concede that the depth of an event is a less plausible observation than a step or a pomset, and to some extent DB and its variants are a technical matter. Nevertheless, it is interesting that observing depth is precisely equivalent to observing steps, when both forward and reverse observations are allowed.

One of the main purposes van Glabbeek and Goltz had in studying bisimulation-based equivalences on stable configuration structures was to identify equivalences that are preserved by the refinement of actions. Of the equivalences we discuss here, they identified two that are preserved by refinement, namely history-preserving bisimulation and hereditary history-preserving bisimulation. We round out their study by showing (Proposition 8.1) that all the reverse-type equivalences considered here fail to be preserved by refinement, with the possible exception of those few that are sandwiched between history-preserving bisimulation and hereditary history-preserving bisimulation; see Figure 24 in Section 10. Thus Proposition 8.1 shows that there is a trade-off between the observational character of an equivalence and its preservation by refinement of actions.

The paper is organised as follows. We start by defining stable configuration structures in Section 2. Then in Section 3 we look at forward-only equivalences, including our new depth-respecting equivalences. Section 4 is devoted to equivalences enhanced with the power of reversing single events. In Section 5 we look at equivalences with reverse steps, and show among other results that these are equivalent to reverse depth-respecting equivalences and that they give the same power as forward steps. We briefly investigate equivalences with reverse pomsets in Section 6, and those with the hereditary property in Section 7. Section 8 briefly discusses which equivalences are preserved under refinement.

In Section 9, we show that Bednarczyk's result described above still holds in the absence of equidepth auto-concurrency, and we look at how the hierarchy of equivalences collapses under this assumption. We then draw some conclusions in Section 10.

2. Configuration structures and prime event structures

In this section we define our computational model, namely configuration structures. We start by defining prime event structures, since this is a better-known formalism, and we shall phrase most of our examples in terms of prime event structures.

2.1. *Prime event structures*

We assume a set of action labels Act, ranged over by $a, b, \ldots$.

Definition 2.1. (Nielsen *et al.* 1981) A (labelled) prime event structure is a 4-tuple $\mathscr{E} = (E, <, \sharp, \ell)$ where

- E is a set of *events*;
- $< \subseteq E \times E$ is an irreflexive partial order (the *causality relation*) such that for any $e \in E$, the set $\{e' \in E : e' < e\}$ is finite;
- $\sharp \subseteq E \times E$ is an irreflexive, symmetric relation (the *conflict relation*) such that if $e_1 < e_2$ and $e_1 \sharp e$, then $e_2 \sharp e$;
- $\ell : E \to \mathsf{Act}$ is the *labelling function.*

As we have defined it, prime event structures have *binary* conflict, though this can be generalised to non-binary conflict, where, say, out of three events, only two can occur (van Glabbeek and Vaandrager 1997).

When drawing diagrams of event structures we shall, as usual, represent the causal relation by arrows and the conflict relation by dotted lines. We shall also suppress the actual events, and just write their labels instead. Thus, if we have two events e_1 and e_2, both labelled with a, we shall denote them both by a in diagrams, or, if we wish to distinguish between them, by a_1 and a_2, respectively. This is justified since all the notions of equivalence we shall discuss depend on the labels of the events, rather than the events themselves.

Definition 2.2. Let $\mathscr{E} = (E, <, \sharp, \ell)$ and let $X \subseteq E$. Then:

- X is *conflict-free* if $\sharp \cap (X \times X) = \varnothing$;
- X is *left-closed* if whenever $e \in X$ and $e' < e$, we have $e' \in X$;
- X is a *configuration* of $\mathscr{E}$ if X is finite, left-closed and conflict-free.

2.2. *Configuration structures*

We define configuration structures much as in van Glabbeek and Goltz (2001), but with the the termination predicate omitted for simplicity – we will stay as close as possible to van Glabbeek and Goltz (2001) in most of the definitions in this section.

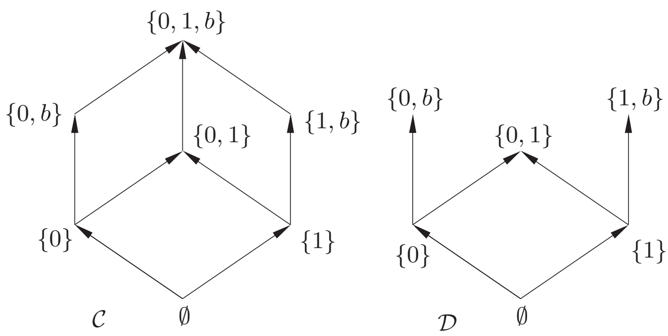

Fig. 1. Non-stable and stable configuration structures.

Definition 2.3. A *configuration structure* (over an alphabet Act) is a pair $\mathscr{C} = (C, \ell)$ where C is a family of finite sets (configurations) and $\ell : \bigcup_{X \in C} X \to \mathsf{Act}$ is a labelling function.

We write $C_{\mathscr{C}}, \ell_{\mathscr{C}}$ to refer to the two components of a configuration structure $\mathscr{C}$. Also, we use $E_{\mathscr{C}} = \bigcup_{X \in C} X$ for the *events* of $\mathscr{C}$. We let $e, \ldots$ range over events, and $E, F, \ldots$ over sets of events.

Definition 2.4 (van Glabbeek and Goltz 2001). A configuration structure $\mathscr{C} = (C, \ell)$ is *stable* if it is:

— rooted – $\emptyset \in C$;
— connected – $\emptyset \neq X \in C$ implies $\exists e \in X : X \setminus \{e\} \in C$;
— closed under bounded unions – if $X, Y, Z \in C$, then $X \cup Y \subseteq Z$ implies $X \cup Y \in C$;
— closed under bounded intersections – if $X, Y, Z \in C$, then $X \cup Y \subseteq Z$ implies $X \cap Y \in C$.

Note that in the presence of the 'closed under bounded unions' condition, the 'closed under bounded intersections' condition can be simplified to 'if $X, Y, X \cup Y \in C$, then $X \cap Y \in C$'.

Any stable configuration structure is the set of configurations of a stable event structure (van Glabbeek and Goltz 2001, Theorem 5.3)†. Figure 1 gives two example configuration structures derived from examples in Winskel (1987). In each of $\mathscr{C}, \mathscr{D}$, the labelling can be taken to be the identity function. Configuration structure $\mathscr{C}$ models a 'parallel switch' where events 0 or 1 can light the bulb b. If both 0 and 1 occur, then b can also happen. This is *inclusive* 'or' causation. We see that $\mathscr{C}$ is not stable since it is not closed under bounded intersections: $\{0, b\} \cup \{1, b\}$ is bounded by $\{0, 1, b\}$, but $\{0, b\} \cap \{1, b\} = \{b\} \notin C_{\mathscr{C}}$. By contrast, $\mathscr{D}$ is stable; it models a switch where the bulb can be lit by either 0 or 1, but not both, that is, *exclusive* 'or' causation.

Configuration structures have associated notions of causal orderings on events and concurrency between events.

† We work with stable configuration structures alone and never directly with stable event structures. The definition of stable event structures can be found in Winskel (1987) and van Glabbeek and Goltz (2001).

Definition 2.5. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure, and let $X \in C$.

— Causality – $d \leqslant_X e$ if and only if for all $Y \in C$ with $Y \subseteq X$ we have $e \in Y$ implies $d \in Y$. Furthermore $d <_X e$ if and only if $d \leqslant_X e$ and $d \neq e$.
— Concurrency – $d\ co_X\ e$ if and only if $d \nless_X e$ and $e \nless_X d$.

Van Glabbeek and Goltz (2001) showed that $<_X$ is a partial order and that the sub-configurations of X are precisely those subsets Y that are left-closed with respect to $<_X$, that is, if $d <_X e \in Y$, then $d \in Y$. Furthermore, if $X, Y \in C$ with $Y \subseteq X$, then $<_Y = <_X \restriction Y$.

It is easy to check that the set of configurations of a prime event structure forms a configuration structure. In fact, prime event structures form a proper subclass of stable event structures; the configuration structures associated with prime event structures are obtained by strengthening the 'closed under bounded intersections' condition of Definition 2.3 to closed under intersections: if $X, Y \in C$, then $X \cap Y \in C$ (van Glabbeek 1996). In Figure 1, $\mathscr{D}$ is not prime, since it is not closed under intersections: $\{0, b\} \cap \{1, b\} = \{b\} \notin C_{\mathscr{D}}$. Thus, prime event structures do not allow 'or' causation; to model the switch as a prime event structure we would have to model the lighting of the bulb as two separate events, one caused by 0 and the other by 1.

3. Forward-only equivalences

In this section we investigate the hierarchy of forward-only bisimulation-based equivalences. We start by recalling existing equivalences (Section 3.1), and then we introduce new depth-respecting and homogeneous equivalences in Sections 3.2 and 3.3. Depth-respecting bisimulation, while of lesser interest in its own right, will help us obtain a simpler proof of Fecher's result that weak history-preserving equivalence is included in SB equivalence. So-called homogeneous variants of step bisimulation are again of lesser interest in their own right, but are needed in later technical developments in Sections 5 and 9.

3.1. *Existing equivalences*

In this section we will recall the definition of various notions of equivalence between configuration structures. We start by defining the most basic labelled transition relation, on single events.

Definition 3.1. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $a \in \mathsf{Act}$. We let $X \xrightarrow{a}_{\mathscr{C}} X'$ if and only if $X, X' \in C$, $X \subseteq X'$ and $X' \setminus X = \{e\}$ with $\ell(e) = a$.

We shall depict many of our example configuration structures as labelled transition diagrams where the nodes are configurations, with the empty configuration as the start node. We will usually not give the actual configurations as sets of events, but it is clear that these can be recovered from such diagrams, up to isomorphism.

Definition 3.2 (van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is an *interleaving bisimulation* (IB) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$, then for $a \in \mathsf{Act}$:

— if $X \xrightarrow{a}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{a}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{a}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are IB equivalent ($\mathscr{C} \approx_{ib} \mathscr{D}$) if and only if there is an IB between $\mathscr{C}$ and $\mathscr{D}$.

For a set of events E, let $\ell(E)$ be the multiset of labels of events in E. We now define a *step* transition relation where concurrent events are executed in a single step.

Definition 3.3. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and assume $A \in \mathbb{N}^{\mathsf{Act}}$ (A is a multiset over Act). We let $X \xrightarrow{A}_{\mathscr{C}} X'$ if and only if $X, X' \in C$, $X \subseteq X'$ and $X' \setminus X = E$ with $d\ co_{X'}\ e$ for all $d, e \in E$ and $\ell(E) = A$.

Definition 3.4 (Pomello 1986; van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is a *step bisimulation* (SB) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$, then for $A \in \mathbb{N}^{\mathsf{Act}}$:

— if $X \xrightarrow{A}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{A}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{A}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{A}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are SB equivalent ($\mathscr{C} \approx_{sb} \mathscr{D}$) if and only if there is an SB between $\mathscr{C}$ and $\mathscr{D}$.

Definition 3.5. Let $\mathscr{X} = (X, <_X, \ell_X)$ and $\mathscr{Y} = (Y, <_Y, \ell_Y)$ be partial orders that are labelled over Act. We say that $\mathscr{X}$ and $\mathscr{Y}$ are *isomorphic* ($\mathscr{X} \cong \mathscr{Y}$, or sometimes just $X \cong Y$) if and only if there is a bijection from X to Y respecting the ordering and labelling. The isomorphism class $[\mathscr{X}]_{\cong}$ of a partial order labelled over Act is called a *pomset* over Act. We let $u, \ldots$ range over pomsets.

Definition 3.6. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and u be a pomset over Act. We let $X \xrightarrow{u}_{\mathscr{C}} X'$ if and only if $X, X' \in C$, $X \subseteq X'$ and $X' \setminus X = H$ with

$$u = [(H, <_{X'} \cap (H \times H), \ell_{\mathscr{C}} \restriction H)]_{\cong}.$$

Definition 3.7 (Boudol and Castellani 1987; van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is a *pomset bisimulation* (PB) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$, then for u any pomset over Act:

— if $X \xrightarrow{u}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{u}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{u}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{u}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are PB equivalent ($\mathscr{C} \approx_{pb} \mathscr{D}$) if and only if there is a PB between $\mathscr{C}$ and $\mathscr{D}$.

Definition 3.8 (De Nicola *et al.* 1987; van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is a *weak history-preserving* (*WH*) *bisimulation* between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$ and $a \in \mathsf{Act}$, then:

— $(X, <_X, \ell_{\mathscr{C}} \restriction X) \cong (Y, <_Y, \ell_{\mathscr{D}} \restriction Y)$;
— if $X \xrightarrow{a}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{a}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{a}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are WH equivalent ($\mathscr{C} \approx_{wh} \mathscr{D}$) if and only if there is a WH bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

We can define a further equivalence by combining pomset and weak history-preserving bisimulation as follows.

Definition 3.9 (van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is a *weak history-preserving pomset bisimulation* (*WHPB*) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$ and u is a pomset over Act, then:

— $(X, <_X, \ell_{\mathscr{C}} \upharpoonright X) \cong (Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$;
— if $X \xrightarrow{u}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{u}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{u}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{u}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are WHPB equivalent ($\mathscr{C} \approx_{whpb} \mathscr{D}$) if and only if there is a WHPB between $\mathscr{C}$ and $\mathscr{D}$.

Definition 3.10 (Rabinovich and Trakhtenbrot 1988; van Glabbeek and Goltz 2001). Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}} \times \mathscr{P}(E_{\mathscr{C}} \times E_{\mathscr{D}})$ is a *history-preserving* (*H*) *bisimulation* between $\mathscr{C}$ and $\mathscr{D}$ if and only if $(\emptyset, \emptyset, \emptyset) \in R$ and if $(X, Y, f) \in R$ and $a \in \mathsf{Act}$, then:

— f is an isomorphism between $(X, <_X, \ell_{\mathscr{C}} \upharpoonright X)$ and $(Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$;
— if $X \xrightarrow{a}_{\mathscr{C}} X'$, then $\exists Y', f'.\ Y \xrightarrow{a}_{\mathscr{D}} Y'$, $(X', Y', f') \in R$ and $f' \upharpoonright X = f$;
— if $Y \xrightarrow{a}_{\mathscr{D}} Y'$, then $\exists X', f'.\ X \xrightarrow{a}_{\mathscr{C}} X'$, $(X', Y', f') \in R$ and $f' \upharpoonright X = f$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are H equivalent ($\mathscr{C} \approx_h \mathscr{D}$) if and only if there is an H bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

Van Glabbeek and Goltz (2001) showed that considering more complex transitions such as step transitions or pomset transitions in the definition of H bisimulation does not change the resulting notion of equivalence.

Proposition 3.11 (van Glabbeek and Goltz 2001). On stable configuration structures,

$$\approx_h \subsetneq \approx_{whpb} \subsetneq \approx_{pb} \subsetneq \approx_{sb} \subsetneq \approx_{ib}$$

and

$$\approx_{whpb} \subsetneq \approx_{wh} \subsetneq \approx_{ib}.$$

We will now give some examples to show that the inclusions in Proposition 3.11 are proper. Here, and subsequently, we will use a CCS-like notation to refer to simple configuration structures.

Example 3.12. IB equivalence is insensitive to auto-concurrency: $a \mid a = a.a$ holds for $\approx_{ib}$, but not for $\approx_{sb}$ or $\approx_{wh}$.

Example 3.13. $a \mid a = (a \mid a) + a.a$ holds for $\approx_{sb}$, but not for $\approx_{pb}$.

Example 3.14 (van Glabbeek and Goltz 2001, Example 9.1). We have

$$a.(b + c) + (a \mid b) + a.b = a.(b + c) + (a \mid b)$$

holds for $\approx_{pb}$, but not for $\approx_{wh}$.

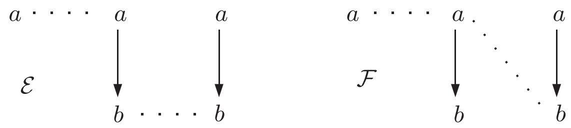

Fig. 2. Example 3.15

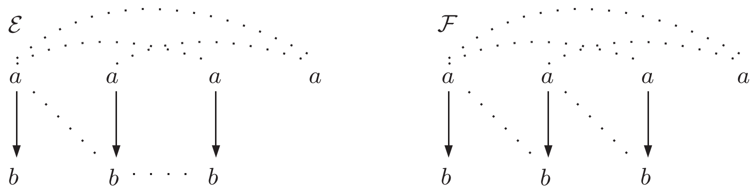

Fig. 3. Example 3.16

Example 3.15 (Bednarczyk 1991, Example 1.6; van Glabbeek and Goltz 2001, Example 9.3). $\mathscr{E}$ and $\mathscr{F}$ are defined as in Figure 2. Recall that we represent the causal relation between events by arrows, and the conflict relation by dotted lines. $\mathscr{E} = \mathscr{F}$ holds for $\approx_{wh}$, but not for $\approx_{pb}$.

Example 3.16 (van Glabbeek and Goltz 2001, Example 9.4, page 294). $\mathscr{E}$ and $\mathscr{F}$ are defined as in Figure 3. We have $\mathscr{E} = \mathscr{F}$ holds for $\approx_{whpb}$, but not for $\approx_h$.

Proposition 3.11 resolves all the implications between the equivalences introduced so far, apart from the relationship between $\approx_{wh}$ and $\approx_{sb}$, which was resolved by Fecher.

Proposition 3.17 (Fecher 2004, Theorem 9). On stable configuration structures,

$$\approx_{wh} \subseteq \approx_{sb}.$$

The inclusion in Proposition 3.17 is proper since we know that $\approx_{sb} \nsubseteq \approx_{wh}$ by Example 3.14.

3.2. *Depth-respecting equivalences*

We will now use the *depth* of an event within a configuration, that is, the length of the longest causal chain leading up to and including the event, as a new notion of an observation that we can make of an event, and study equivalences based on this notion. It will allow us to give a much shorter and more straightforward proof of Proposition 3.17, and will be the vital technique in proving results about reverse step equivalences in Section 5.

We start by defining depth.

Definition 3.18. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure, and let $X \in C$, $e \in X$. The *depth* of e with respect to X (and implicitly $\mathscr{C}$) is given by

$$\mathsf{depth}_X(e) \stackrel{\text{df}}{=} \begin{cases} 1 & \text{if } e \text{ is minimal in } X \\ \max\{\mathsf{depth}_X(e') : e' <_X e\} + 1 & \text{otherwise.} \end{cases}$$

The depth of an event e is the length of the longest causal chain in X up to and including e. Clearly, if $e <_X e'$, then $\mathsf{depth}_X(e) < \mathsf{depth}_X(e')$. Note that if $X, Y \in C$ and $e \in X \cap Y$, then it is not necessarily the case that $\mathsf{depth}_X(e) = \mathsf{depth}_Y(e)$ (due to the fact that a single event can have different possible sets of causes). However, if $X \cup Y \subseteq Z$ for some $Z \in C$, then $\mathsf{depth}_X(e) = \mathsf{depth}_Y(e)$.

Definition 3.19. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $a \in \mathsf{Act}$, $k \in \mathbb{N}$. We let $X \xrightarrow{a,k}_{\mathscr{C}} X'$ if and only if $X, X' \in C$, $X \subseteq X'$ and $X' \setminus X = \{e\}$ with $\ell(e) = a$, $\mathsf{depth}_{X'}(e) = k$.

Definition 3.20. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is a *depth-respecting* bisimulation (DB) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset) \in R$ and if $(X, Y) \in R$, then for $a \in \mathsf{Act}$ and $k \in \mathbb{N}$:

— if $X \xrightarrow{a,k}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \xrightarrow{a,k}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \xrightarrow{a,k}_{\mathscr{D}} Y'$, then $\exists X'.\ X \xrightarrow{a,k}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are DB equivalent ($\mathscr{C} \approx_{db} \mathscr{D}$) if and only if there is a DB between $\mathscr{C}$ and $\mathscr{D}$.

Proposition 3.21. On stable configuration structures, $\approx_{db} \subsetneq \approx_{sb}$.

Proof. Suppose $\mathscr{C} \approx_{db} \mathscr{D}$ through DB R. We show that R is an SB. Let $A \in \mathbb{N}^{\mathsf{Act}}$, and suppose $R(X, Y)$.

Assume $X \xrightarrow{A}_{\mathscr{C}} X'$. Let $E = X' \setminus X$ and $\{e_1, \ldots, e_n\}$ be an enumeration of E in non-increasing order of depth with respect to X', that is, letting $\mathsf{depth}_{X'}(e_i) = k_i$, we have $k_i \geqslant k_j$ for $i < j \leqslant n$. Let $\ell_{\mathscr{C}}(e_i) = a_i$ $(i \leqslant n)$. Then

$$X = X_0 \xrightarrow{a_1,k_1}_{\mathscr{C}} X_1 \cdots \xrightarrow{a_n,k_n}_{\mathscr{C}} X_n = X'.$$

So

$$Y = Y_0 \xrightarrow{a_1,k_1}_{\mathscr{D}} Y_1 \cdots \xrightarrow{a_n,k_n}_{\mathscr{D}} Y_n = Y'$$

for some $Y_1, \ldots, Y_n$ such that $R(X_i, Y_i)$ $(i \leqslant n)$. Let $e'_i = Y_i \setminus Y_{i-1}$ $(i = 1, \ldots, n)$. The e'_i must all be pairwise concurrent: if $i < j$, then $e'_j <_{Y'} e'_i$ is impossible since $e'_j \notin Y_i$ and Y_i is left-closed; also, $e'_i <_{Y'} e'_j$ is impossible since $\mathsf{depth}_{Y'}(e_i) \geqslant \mathsf{depth}_{Y'}(e_j)$. Hence, $Y \xrightarrow{A}_{\mathscr{D}} Y'$ with $R(X', Y')$, as required.

By symmetry, we also have that if $Y \xrightarrow{A}_{\mathscr{C}} Y'$, then $X \xrightarrow{A}_{\mathscr{C}} X'$ for some X' such that $R(X', Y')$. This shows that $\approx_{db} \subseteq \approx_{sb}$.

The inclusion is proper by Example 3.13 since $a|a = (a|a) + a.a$ holds for $\approx_{sb}$, but not for $\approx_{db}$. □

Example 3.22. In this example, we modify van Glabbeek and Goltz (2001, Example 9.3) by relabelling the last a (reading from left to right) as a' to get $\mathscr{E}$ and $\mathscr{F}$ as in Figure 4. This removes auto-concurrency. The modified structures are not PB-equivalent since in the left-hand one, necessarily after a we can always do pomset $a' < b$, but not in the right-hand one. It is not hard to see that they are not WH-equivalent either. Also, the modified structures are DB-equivalent.

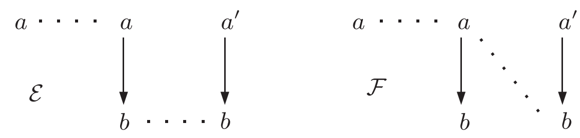

Fig. 4. Example 3.22.

The equivalences $\approx_{pb}$ and $\approx_{db}$ are incomparable: $\approx_{db} \nsubseteq \approx_{pb}$ by Example 3.22, and $\approx_{pb} \nsubseteq \approx_{db}$ by Example 3.14, since this fails to hold for $\approx_{db}$ as well as $\approx_{wh}$.

Lemma 3.23. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Let $X \in C_{\mathscr{C}}, Y \in C_{\mathscr{D}}$. Suppose

$$(X, <_X, \ell_{\mathscr{C}} \upharpoonright X) \cong (Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$$

through isomorphism f. Then for any $e \in X$, we have $\mathsf{depth}_X(e) = \mathsf{depth}_Y(f(e))$.

Proposition 3.24. On stable configuration structures, $\approx_{wh} \subsetneq \approx_{db}$.

Proof. Suppose $\mathscr{C} \approx_{db} \mathscr{D}$ through WH bisimulation R. We will show that R is a DB. Suppose $R(X, Y)$ and $X \xrightarrow{a,k}_{\mathscr{C}} X'$. Since R is a WH bisimulation, there is Y' such that $Y \xrightarrow{a}_{\mathscr{C}} Y'$ and $R(X, Y)$. We know $X \cong Y$, so for each $k' \geqslant 0$, using Lemma 3.23, X and Y have the same number of events with depth k'. The same is true for X' and Y', since $X' \cong Y'$. Let $X' \setminus X = \{e\}$ and $Y' \setminus Y = \{e'\}$. We have $\mathsf{depth}_{X'}(e) = k$. Since Y' has one more event with depth k than Y, and for any $k' \neq k$, Y' has the same number of events of depth k' as Y, we deduce that $\mathsf{depth}_{Y'}(e') = k$. Hence $Y \xrightarrow{a,k}_{\mathscr{D}} Y'$, as required.

The case when $Y \xrightarrow{a,k}_{\mathscr{D}} Y'$ is similar. Finally, the inclusion is proper by Example 3.22. □

Combining Propositions 3.24 and 3.21, we obtain an alternative proof of Proposition 3.17 that is much shorter and more straightforward than the original proof in Fecher (2004).

We show the inclusions between all equivalences introduced so far in Figure 5. All inclusions are proper, and no other inclusions hold, as we have seen using various counterexamples.

3.3. *Homogeneous and equidepth versions*

In this section we define and briefly discuss three minor variants of step bisimulation. They will be vital in the proofs of results in Section 5 and also have an impact on the developments in Section 9, where we improve on Bednarczyk's characterisation of H-H in the setting with no equidepth auto-concurrency.

We will say that a set of events is *homogeneous* if all events have the same label. Similarly, a multiset of labels is homogeneous if all labels are the same. A step transition $X \xrightarrow{A}_{\mathscr{C}} X'$ is homogeneous if A is homogeneous. We can define *homogeneous step bisimulation* (HSB) and the associated equivalence $\approx_{hsb}$ by modifying step bisimulation (Definition 3.4) by requiring all step transitions to be homogeneous.

We define the *equidepth* step transition relation as follows.

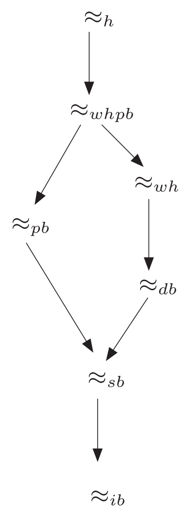

Fig. 5. Equivalences discussed so far.

Definition 3.25. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $A \in \mathbb{N}^{\mathsf{Act}}$. We let $X \xrightarrow{A,=}_{\mathscr{C}} X'$ if and only if $X, X' \in C$, $X \subseteq X'$ and $X' \setminus X = E$ with $\mathsf{depth}_{X'}(d) = \mathsf{depth}_{X'}(e)$ for all $d, e \in E$ and $\ell(E) = A$.

It is immediate that if $X \xrightarrow{A,=}_{\mathscr{C}} X'$, then $X \xrightarrow{A}_{\mathscr{C}} X'$ in the usual sense.

We can then define:

— *equidepth step bisimulation* and $\approx_{esb}$
— *homogeneous equidepth step bisimulation* and $\approx_{hesb}$

in an obvious fashion.

The equation $a \,|\, a = a.a$ (Example 3.12) holds for $\approx_{ib}$ but not for $\approx_{hsb}$ or $\approx_{hesb}$.

Example 3.26. For $a \neq b$, we have $a \,|\, b = a.b + b.a$ holds for $\approx_{hsb}$ and $\approx_{hesb}$, but not for $\approx_{sb}$ or $\approx_{esb}$.

Example 3.27. $a \,|\, (b.a) = (a \,|\, b).a$ (where $a \neq b$) holds for $\approx_{esb}$, but not for $\approx_{hsb}$.

Proposition 3.28. On stable configuration structures:

(1) $\approx_{db} \subsetneq \approx_{esb} \subsetneq \approx_{hesb} \subsetneq \approx_{ib}$;
(2) $\approx_{sb} \subsetneq \approx_{hsb} \subsetneq \approx_{ib}$.

Proof. It is straightforward to show the inclusions. They are proper by Examples 3.12, 3.26 and 3.27. □

The next example shows that $\approx_{sb} \nsubseteq \approx_{hesb}$.

Example 3.29. $a \,|\, (a.a) + a.(a \,|\, a) = a \,|\, (a.a)$ holds for $\approx_{sb}$, but not for $\approx_{hesb}$.

Since Example 3.29 does not hold for $\approx_{pb}$, we need a different example to show that $\approx_{pb} \nsubseteq \approx_{hesb}$.

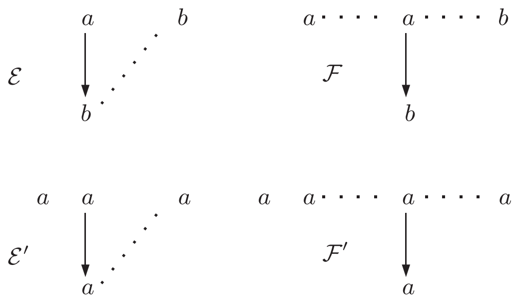

Fig. 6. Example 3.30.

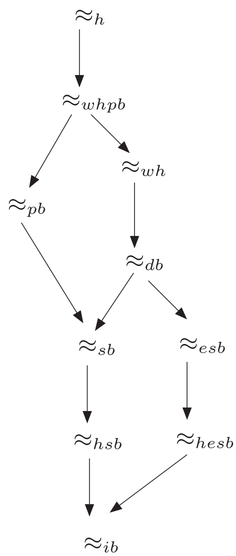

Fig. 7. Adding the homogeneous and equidepth step equivalences.

Example 3.30. The first pair of structures $\mathscr{E}, \mathscr{F}$ in Figure 6 are PB- and ESB-equivalent, but not DB-equivalent – this example is taken from van Glabbeek and Goltz (1989). We modify $\mathscr{E}, \mathscr{F}$ by letting all events have the same label a and adding a new event in parallel, also labelled with a. This yields $\mathscr{E}', \mathscr{F}'$ as shown also in Figure 6. Let $R = \{(X, Y) : X \in C_{\mathscr{E}'}, Y \in C_{\mathscr{F}'}, |X| = |Y|\}$. One can check that R is a PB, so $\mathscr{E}' \approx_{pb} \mathscr{F}'$. However, $\mathscr{E}' \not\approx_{hesb} \mathscr{F}'$. This is because in $\mathscr{E}'$, after performing a, an equidepth $\{a, a\}$ transition is always possible, while in $\mathscr{F}'$ after performing the a in conflict with two other events, no such equidepth $\{a, a\}$ transition is possible.

Figure 7 shows the inclusions between all the equivalences introduced so far. All inclusions are proper. Any inclusions that are not shown do not hold, as demonstrated by the various examples we have presented.

4. Reverse single-event transitions

In this section we explore the hierarchy of equivalences when we add the power of reverse computation with single-event transitions.

We use a reverse transition relation, represented by a wavy arrow, which simply inverts the standard forward version.

Definition 4.1. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $a \in \mathsf{Act}$. Then $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$ if and only if $X' \overset{a}{\rightarrow}_{\mathscr{C}} X$.

We can now define reverse single-event versions of all the equivalences defined in Section 3. Let FB stand for any of the forward-only bisimulations $\approx_{fb}$ defined in Section 3, apart from history-preserving (H) bisimulation. Thus FB may be any one of IB, SB, PB, WH, WHPB, DB, and so on.

Definition 4.2. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $a \in \mathsf{Act}$. Then $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}}$ is an RI-FB if and only if R is an FB and if $R(X, Y)$, then for any $a \in \mathsf{Act}$:

— if $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$, then $\exists Y'.\ Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$ and $(X', Y') \in R$;
— if $Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$, then $\exists X'.\ X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$ and $(X', Y') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are RI-FB equivalent ($\mathscr{C} \approx_{ri\text{-}fb} \mathscr{D}$) if and only if there is an RI-FB bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

Definition 4.3. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structures and let $a \in \mathsf{Act}$. Then $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}} \times \mathscr{P}(E_{\mathscr{C}} \times E_{\mathscr{D}})$ is an RI-H bisimulation if and only if R is an H bisimulation and if $R(X, Y, f)$, then for any $a \in \mathsf{Act}$:

— if $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$, then $\exists Y', f'.\ Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$ and $(X', Y', f') \in R$;
— if $Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$, then $\exists X', f'.\ X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$ and $(X', Y', f') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are RI-H equivalent ($\mathscr{C} \approx_{ri\text{-}h} \mathscr{D}$) if and only if there is an RI-H bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

Notice that Definition 4.3 does not assert any relationship between f' and f.

Remark 4.4. What we here call $\approx_{ri\text{-}ib}$ was previously defined in Bednarczyk (1991), where it was called *back & forth* bisimulation ($\sim_{b\&f}$); we called it forward–reverse (FR) bisimulation in Phillips and Ulidowski (2007b) and reverse bisimulation (RB) in Phillips and Ulidowski (2010).

De Nicola, Montanari and Vaandrager also investigated 'back & forth' bisimulations (De Nicola *et al.* 1990; De Nicola and Vaandrager 1990), but their relations were defined over computations (paths) rather than states (for example, in the process $a \mid b$, after performing a followed by b, one can only reverse immediately on b, and not a). As a result, in the absence of τ actions, the distinguishing power of these bisimulations is that of IB (De Nicola *et al.* 1990), and hence it is less than that of RI-IB.

As far as we are aware, $\approx_{ri\text{-}fb}$ equivalences other than $\approx_{ri\text{-}ib}$ have not been investigated previously.

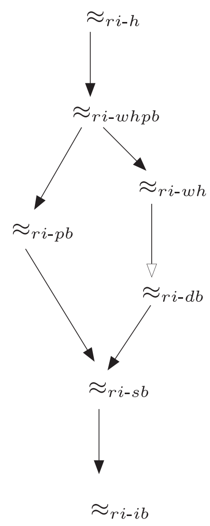

Fig. 8. Equivalences with reverse single-event transitions. Arrows with solid heads represent proper inclusions; the arrow with an open head represents an inclusion that is not known to be proper.

Given any two forward-only equivalences $\approx_{fb}$, $\approx_{fb'}$, it is clear from the definitions that if $\approx_{fb} \subseteq \approx_{fb'}$, then $\approx_{ri\text{-}fb} \subseteq \approx_{ri\text{-}fb'}$. It is also immediate from the definitions that for any forward-only equivalence $\approx_{fb}$, we have $\approx_{ri\text{-}fb} \subseteq \approx_{fb}$.

Figure 8 shows the inclusions between all the $\approx_{ri\text{-}fb}$ equivalences. We now give various examples to show that the inclusions denoted by arrows with solid heads are proper:

— The equation $a \mid a = a.a$ (Example 3.12) holds for $\approx_{ri\text{-}ib}$ but not for $\approx_{sb}$, and therefore not for $\approx_{ri\text{-}sb}$.

— The equation $a \mid a = (a \mid a) + a.a$ (Example 3.13) holds for $\approx_{ri\text{-}sb}$, but not for $\approx_{pb}$ or $\approx_{db}$, and therefore not for $\approx_{ri\text{-}pb}$ or $\approx_{ri\text{-}db}$.

Both examples just cited use auto-concurrency, and we see that the extra power of reverse single-event transitions makes no difference in these two cases.

When it comes to distinguishing $\approx_{ri\text{-}h}$, $\approx_{ri\text{-}whpb}$, $\approx_{ri\text{-}wh}$ and $\approx_{ri\text{-}pb}$, we cannot use Examples 3.14, 3.15 and 3.16 as we did in the forward-only case, since all three examples are invalid for $\approx_{ri\text{-}ib}$. We have not been able to locate any examples in the literature that serve the purpose, so we will present three new examples (Examples 4.5, 4.7 and 4.8).

Example 4.5. Figure 9 shows the transition systems for structures $\mathscr{E}$, $\mathscr{F}$. We have $\mathscr{E} = \mathscr{F}$ holds for $\approx_{ri\text{-}pb}$, but not for $\approx_{ri\text{-}db}$, and hence not for $\approx_{ri\text{-}wh}$. Notice that this example uses non-binary conflict between the three initial a-events.

Example 4.6 provides a simpler alternative to Example 4.5.

Example 4.6. Figure 10 shows the transition systems for structures $\mathscr{E}$, $\mathscr{F}$.
We have $\mathscr{E} = \mathscr{F}$ holds for $\approx_{ri\text{-}pb}$, but not for $\approx_{db}$, and hence not for $\approx_{ri\text{-}db}$ or $\approx_{ri\text{-}wh}$.

Example 4.7. In this example, we adapt Example 3.15 (which was suggested to van Glabbeek and Goltz by Rabinovich) by adding a new event (b_1), to obtain the structures

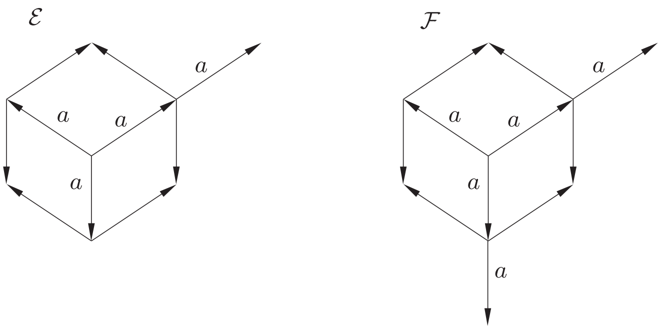

Fig. 9. Example 4.5.

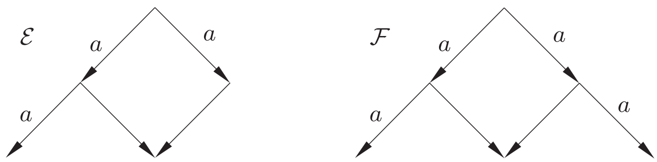

Fig. 10. Example 4.6.

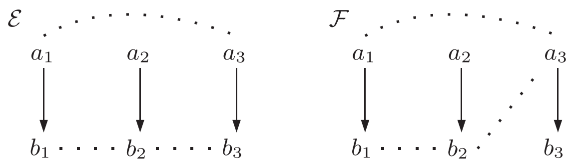

Fig. 11. Example 4.7.

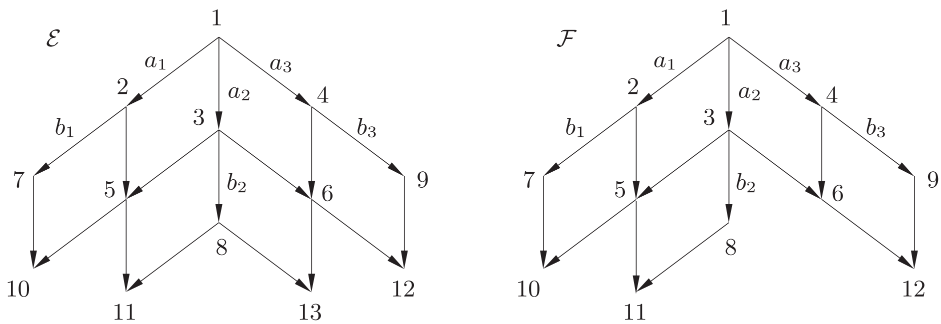

Fig. 12. Transition diagram for Example 4.7.

$\mathscr{E}$, $\mathscr{F}$ shown in Figure 11. Here, as elsewhere, when we label events as $a_1, a_2, \ldots$ we mean that there are distinct events $e_1, e_2, \ldots$ that are labelled with a. Then $\mathscr{E} \not\approx_{pb} \mathscr{F}$ by the same reasoning as in van Glabbeek and Goltz (2001): in $\mathscr{E}$, we can perform pomset transition $a \to b$ after any a, but in $\mathscr{F}$, we cannot perform $a \to b$ after a_3 – see Figure 12.

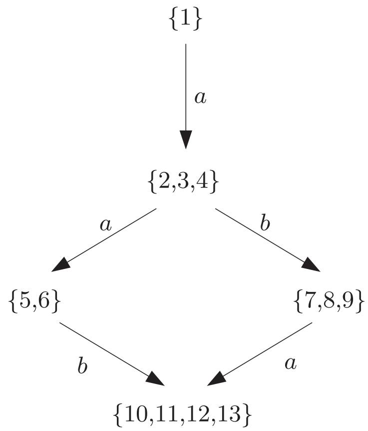

Fig. 13. Bisimulation for Example 4.7.

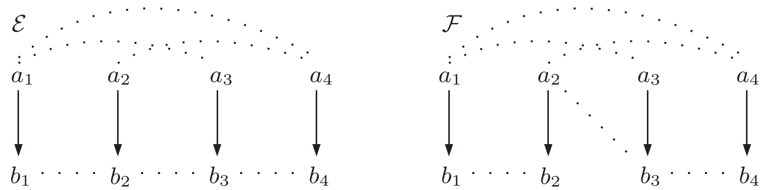

Fig. 14. Example 4.8.

We will now check that $\mathscr{E} \approx_{ri\text{-}wh} \mathscr{F}$. The only difference is that configuration 13 ($\{a_2, a_3, b_3\}$) is missing in $\mathscr{F}$. We define $R = \{(X, Y) : X \in C_{\mathscr{E}}, Y \in C_{\mathscr{F}}, X \cong Y\}$ and check that R is an RI-WH bisimulation – see the transition diagram in Figure 13. The meaning is that all configurations in each source set can perform a transition to at least one member of the target set. Also, each configuration in the target set can perform a reverse transition to at least one member of the source set. This works for both $\mathscr{E}$ and $\mathscr{F}$, showing that $\mathscr{E} \approx_{ri\text{-}wh} \mathscr{F}$.

Example 4.8. In this example, we adapt Example 3.16 by adding a new event (b_4) and changing the conflict between a_1 and b_2 to be between b_1 and b_2, and obtain the structures $\mathscr{E}$, $\mathscr{F}$ shown in Figure 14. We have $\mathscr{E} = \mathscr{F}$ holds for $\approx_{ri\text{-}whpb}$, but not for $\approx_h$, and hence not for $\approx_{ri\text{-}h}$. We will check that $\mathscr{E} \approx_{ri\text{-}whpb} \mathscr{F}$. The transition diagrams are shown in Figure 15. The only difference is that configuration 18 ($\{a_2, a_3, b_3\}$) is missing in $\mathscr{F}$. We will now define a bisimulation by relating all isomorphic states, and check that it is an RI-WHPB – see the transition diagram in Figure 16. All pomset transitions work equally well for $\mathscr{E}$ and $\mathscr{F}$ in both directions.

To see that $\mathscr{E}$ and $\mathscr{F}$ are not H-equivalent, consider $1 \xrightarrow{a_2} \xrightarrow{a_3} 7$ in $\mathscr{F}$. This must be matched by $1 \xrightarrow{a_i} \xrightarrow{a_j} \{a_i, a_j\}$ in $\mathscr{E}$. Here $\{a_i, a_j\}$ must be one of 6, 7 or 8. But then both b_i and b_j are possible. However, 7 in $\mathscr{F}$ can only do b_2. Hence, one of the b_i and b_j transitions in $\mathscr{E}$ cannot be matched. (We have to stick to the isomorphism already established for a_2, a_3 to be history-preserving.)

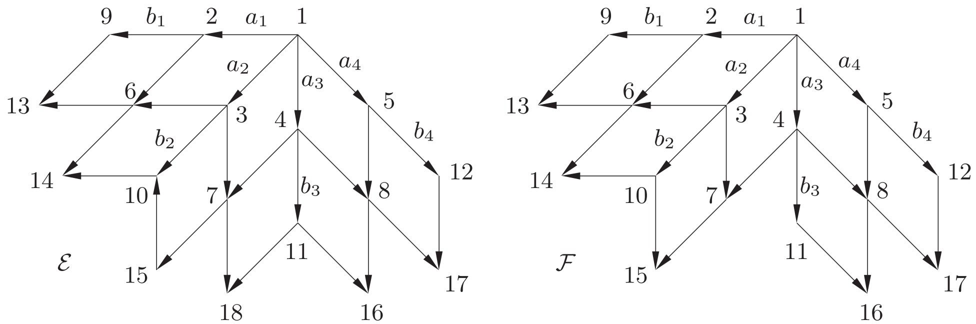

Fig. 15. Transition diagram for Example 4.8.

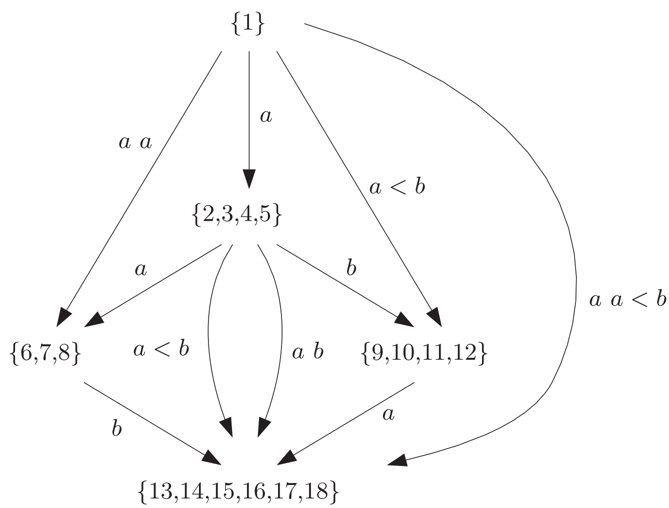

Fig. 16. Bisimulation for Example 4.8.

We have now shown that all the inclusions in Figure 8 represented by arrows with solid heads are proper. There is one arrow with an open head from $\approx_{ri\text{-}wh}$ to $\approx_{ri\text{-}db}$, and whether this inclusion is proper is still an open question. All other inclusions are ruled out by the various examples we have presented.

We now turn to the question of relationships between the forward-only equivalences of Figure 7 and the reverse single-event versions of Figure 8. We have already seen that for every forward-only equivalence FB, $\approx_{ri\text{-}fb} \subsetneq \approx_{fb}$. These latter inclusions are all proper as shown by the next example, which shows that the simplest reverse equivalence $\approx_{ri\text{-}ib}$ has extra power not available to the strongest forward-only equivalence $\approx_h$.

Example 4.9. The Absorption Law (Boudol and Castellani 1987; Bednarczyk 1991; van Glabbeek and Goltz 2001)

$$(a\,|\,(b+c)) + (a\,|\,b) + ((a+c)\,|\,b) = (a\,|\,(b+c)) + ((a+c)\,|\,b)$$

holds for $\approx_h$, but not for $\approx_{ri\text{-}ib}$.

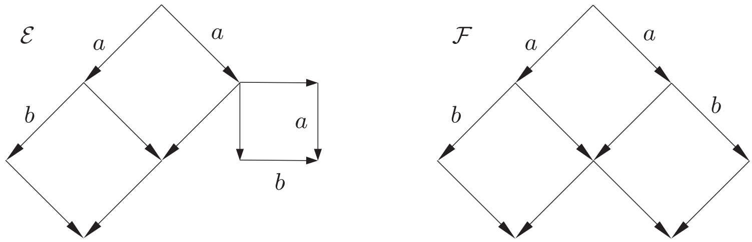

Fig. 17. An example where $\mathscr{E} \approx_{ri\text{-}hesb} \mathscr{F}$ but $\mathscr{E} \not\approx_{ri\text{-}esb} \mathscr{F}$.

Another example, which holds for $\approx_h$, but not for $\approx_{ri\text{-}ib}$ appears as Nielsen and Clausen (1994, Example 1).

We will now show that no other inclusions from the reverse single-event equivalences to the forward-only equivalences are possible, except, possibly, $\approx_{ri\text{-}db} \subsetneq \approx_{wh}$. This can be done using the examples already introduced:

— $\approx_{ri\text{-}whpb} \nsubseteq \approx_h$ by Example 4.8.
— One can check that in Example 3.30, $\mathscr{E}' \approx_{ri\text{-}pb} \mathscr{F}'$ using the same bisimulation R already given. Hence $\approx_{ri\text{-}pb} \nsubseteq \approx_{hesb}$.
— $\approx_{ri\text{-}wh} \nsubseteq \approx_{pb}$ by Example 4.7.
— $\approx_{ri\text{-}ib} \nsubseteq \approx_{hsb}$, $\approx_{ri\text{-}ib} \nsubseteq \approx_{hesb}$ by Example 3.12.

Remark 4.10. We could also consider reverse single-event versions of $\approx_{hsb}$, $\approx_{esb}$ and $\approx_{hesb}$. Example 3.12 holds for $\approx_{ri\text{-}ib}$, and fails for $\approx_{hsb}$ and $\approx_{hesb}$, as we have already seen. Therefore $\approx_{ri\text{-}hsb}$ and $\approx_{ri\text{-}hesb}$ are strictly included in $\approx_{ri\text{-}ib}$. Furthermore, Example 3.13 holds for $\approx_{ri\text{-}esb}$, and fails for $\approx_{db}$. Hence, $\approx_{ri\text{-}db}$ is strictly included in $\approx_{ri\text{-}esb}$. A different example shows that $\approx_{ri\text{-}esb}$ is strictly included in $\approx_{ri\text{-}hesb}$ – see Figure 17. However, some open questions remain, in particular, is $\approx_{ri\text{-}sb}$ more powerful than $\approx_{ri\text{-}hsb}$?

5. Reverse steps and depth-preserving transitions

In this section we prove results about reverse step equivalences and reverse depth-preserving equivalences. We show in Section 5.1 that reverse steps are strictly more discriminating than forward steps, and in Section 5.2 we show that reverse depth-preserving equivalences have the same power as reverse step equivalences – the technical developments needed to show this will also be needed in Section 9. In Section 5.3 we put together the results of Sections 5.1 and 5.2, and map out the resulting hierarchy of equivalences.

5.1. *Reverse step equivalences*

As with single events, we use a reverse step transition relation, represented by a wavy arrow, which simply inverts the standard forward step version.

Definition 5.1. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $A \in \mathbb{N}^{\mathsf{Act}}$. Then $X \overset{A}{\rightsquigarrow}_{\mathscr{C}} X'$ if and only if $X' \overset{A}{\rightarrow}_{\mathscr{C}} X$.

In much the same way as for single event reverse transitions, we can now define reverse step versions $\approx_{rs\text{-}fb}$ and $\approx_{rs\text{-}h}$ of all the equivalences defined in Section 3 – we will not give the definitions explicitly here as they are similar to Definitions 4.2 and 4.3. We can also define equivalences $\approx_{rhs\text{-}fb}$ in a corresponding way.

Remark 5.2. What we call $\approx_{rs\text{-}sb}$ here was briefly mentioned in Bednarczyk (1991), where it was called *multi-step back & forth bisimulation* ($\sim_{\mu b\&f}$); we called it *reverse step bisimulation* (*RSB*) in Phillips and Ulidowski (2010).

Recall the notion of a homogeneous forward step from Section 3.3. There we saw that homogeneous forward steps give less power than full forward steps: $\approx_{hsb} \subsetneq \approx_{sb}$. We now show that reverse homogeneous steps give as much power as full reverse steps, and that, in fact, they also have the power of forward steps: we prove $\approx_{rhs\text{-}ib} = \approx_{rs\text{-}sb}$ (Theorem 5.10).

We have seen that $a \,|\, a = (a \,|\, a) + a.a$ (Example 3.13) holds for $\approx_{ri\text{-}sb}$. However, it does not hold for $\approx_{rs\text{-}ib}$. Using Theorem 5.10, this means that $\approx_{ri\text{-}sb}$ is strictly weaker than $\approx_{rs\text{-}ib}$. Thus reverse steps have greater power than forward steps.

Clearly, any RS-SB is an RHS-SB. We shall show the converse also holds, so $\approx_{rhs\text{-}ib} = \approx_{rs\text{-}sb}$ (Theorem 5.10).

First we need some lemmas.

Lemma 5.3. Let $\mathscr{C}$ and $\mathscr{D}$ be stable configuration structures and R be an RI-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then $\ell(X) = \ell(Y)$.

Proof. Suppose $R(X, Y)$ and that $\ell(X) = A$. By connectedness, there are transitions $X \overset{a_1}{\rightsquigarrow}_{\mathscr{C}} \cdots \overset{a_n}{\rightsquigarrow}_{\mathscr{C}} \emptyset$ with $A = \{a_1, \ldots, a_n\}$. Therefore $Y \overset{a_1}{\rightsquigarrow}_{\mathscr{D}} \cdots \overset{a_n}{\rightsquigarrow}_{\mathscr{D}} Y'$ for some Y'. Hence, $\ell(X) \subseteq \ell(Y)$. By symmetrical reasoning, we also have $\ell(Y) \subseteq \ell(X)$. □

Note that Lemma 5.3 would not hold for IBs (or indeed SBs or PBs): for instance, there is an IB between $a + b$ and itself that, while necessarily including the identity relation on configurations, also relates the configuration resulting after performing (the event labelled) a with the configuration after performing b.

For any configuration X, let $\min(X)$ denote its set of minimal elements (with respect to $<_X$); $\min(X)$ is, of course, also a configuration since it is a left-closed subset of X. Note that if X, Y are configurations and $X \subseteq Y$, then $\min(X) \subseteq \min(Y)$.

Lemma 5.4. Let stable configuration structures $\mathscr{C}, \mathscr{D}$ be related by RHS-IB R. Then, if $R(X, Y)$, we have $R(\min(X), \min(Y))$.

Proof. Suppose $R(X, Y)$. Then there are $a_1, \ldots, a_n$ such that $X \overset{a_1}{\rightsquigarrow} \cdots \overset{a_n}{\rightsquigarrow} \min(X)$. Let Y' be such that $Y \overset{a_1}{\rightsquigarrow} \cdots \overset{a_n}{\rightsquigarrow} Y'$ and $R(\min(X), Y')$. Then $\ell(\min(X)) = \ell(Y')$ by Lemma 5.3. We will now show that $Y' = \min(Y)$.

We use the reverse homogeneous steps. Let $A = \ell(\min(X))$. Take any $a \in A$, and let A_a be the multiset of as in A. Then $\min(X) \overset{b_1}{\rightsquigarrow} \cdots \overset{b_n}{\rightsquigarrow} \overset{A_a}{\rightsquigarrow} \emptyset$ for some $b_1, \ldots, b_n \neq a$. Hence, $Y' \overset{b_1}{\rightsquigarrow} \cdots \overset{b_n}{\rightsquigarrow} \overset{A_a}{\rightsquigarrow} \emptyset$. This tells us that all events labelled with a in Y' are minimal, so all events in Y' are minimal since a was arbitrary. So $Y' \subseteq \min(Y)$. Hence $\ell(\min(X)) \subseteq$

$\ell(\min(Y))$. By symmetrical reasoning, we can also establish $\ell(\min(Y)) \subseteq \ell(\min(X))$. So $\ell(\min(Y)) = \ell(\min(X))$. Hence $Y' = \min(Y)$ and $R(\min(X), \min(Y))$, as required. □

We now define the 'lifting' of a configuration structure with respect to a configuration M.

Definition 5.5. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $M \in C$. Define $\mathscr{C}_M = (C_M, \ell_M)$ where

$$C_M = \{X \setminus M : M \subseteq X \in C, \min(X) = \min(M)\}$$
$$\ell_M = \ell \restriction \bigcup_{Y \in C_M} Y.$$

Lemma 5.6. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $M \in C$. Then $\mathscr{C}_M$ is a stable configuration structure.

Lemma 5.7. Let stable configuration structures $\mathscr{C}, \mathscr{D}$ be related by RHS-IB R and let $M \in C_{\mathscr{C}}$ be such that $\min(M) = M$, and, similarly, let $N \in C_{\mathscr{D}}$ be such that $\min(N) = N$. Suppose also that $R(M, N)$. If we define $R_{M,N}$ by

$$R_{M,N} = \{(X \setminus M, Y \setminus N) : R(X, Y),\ \min(X) = M,\ \min(Y) = N\},$$

then $R_{M,N}$ is an RHS-IB between $\mathscr{C}_M$ and $\mathscr{D}_N$.

Proof. Write $R_{M,N}$ as R' for short. We certainly have $R'(\emptyset, \emptyset)$ since $R(M, N)$. Suppose $R'(X \setminus M, Y \setminus N)$ with $R(X, Y)$, $\min(X) = M$ and $\min(Y) = N$.

— Forwards direction:
Suppose $X \setminus M \xrightarrow{a}_{\mathscr{C}_M} X' \setminus M$, where $X' \in C_{\mathscr{C}}$ and $\min(X') = M$. Then $X \xrightarrow{a}_{\mathscr{C}} X'$. Hence there is Y' such that $Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $R(X', Y')$. We need to show that $\min(Y') = N$. Using Lemma 5.4, we have $R(M, \min(Y'))$. By Lemma 5.3, and since $R(M, N)$, we have $\ell(M) = \ell(N)$. Also, since $R(M, \min(Y'))$, we have $\ell(M) = \ell(\min(Y'))$. So $\ell(N) = \ell(\min(Y'))$. Since $N = \min(Y) \subseteq \min(Y')$, we deduce $\min(Y') = N$. So we get $Y \setminus N \xrightarrow{a}_{\mathscr{D}_N} Y' \setminus N$ and $R'(X' \setminus M, Y' \setminus N)$, as required.

— Reverse direction:
Suppose $X \setminus M \overset{A}{\rightsquigarrow}_{\mathscr{C}_M} X' \setminus M$, where $X' \in C_{\mathscr{C}}$ and $\min(X') = M$. Then $X \overset{A}{\rightsquigarrow}_{\mathscr{C}} X'$. Hence there is Y' such that $Y \overset{A}{\rightsquigarrow}_{\mathscr{D}} Y'$ and $R(X', Y')$. We need to know that $\min(Y') = N$. By Lemma 5.4, we have $R(M, \min(Y'))$. So, by Lemma 5.3, $\ell(\min(Y')) = \ell(M) = \ell(N)$. Since $Y' \subseteq Y$, we have $\min(Y') \subseteq \min(Y) = N$, so $\min(Y') = N$. So we get $Y \setminus N \overset{A}{\rightsquigarrow}_{\mathscr{D}_N} Y' \setminus N$ and $R'(X' \setminus M, Y' \setminus N)$, as required. □

Lemma 5.8. Let stable configuration structures $\mathscr{C}, \mathscr{D}$ be related by RHS-IB R and suppose $R(X, Y)$ and $X \xrightarrow{A}_{\mathscr{C}} X'$. Then there is Y' such that $Y \xrightarrow{A}_{\mathscr{D}} Y'$ and $R(X', Y')$.

Proof. The proof is partially inspired by Fecher's proof that a weak history-preserving bisimulation is a step bisimulation (Fecher 2004).

We proceed by induction on $|X'|$:

— Base case: $|X'| = 0$.
$Y' = \emptyset$ will do, trivially.

— Induction step.
Notice that $\min(X) \subseteq \min(X')$. There are two cases:

(1) There is $e \in \min(X') \setminus \min(X)$.
Let $\ell(e) = a$. Then $X \xrightarrow{A\setminus\{a\}} X' \setminus \{e\}$. By induction, there is Y'' such that $Y \xrightarrow{A\setminus\{a\}} Y''$ and $R(X' \setminus \{e\}, Y'')$. Now $X' \setminus \{e\} \xrightarrow{a} X'$. So there is Y' such that $Y'' \xrightarrow{a} Y'$ and $R(X', Y')$. Let e' be the single element in $Y' \setminus Y''$. Now

$$\min(X' \setminus \{e\}) = \min(X') \setminus \{e\}$$

(and $e \in \min(X')$), so

$$|\min(X' \setminus \{e\})| < |\min(X')|.$$

By Lemmas 5.4 and 5.3, we have

$$\ell(\min(X' \setminus \{e\})) = \ell(\min(Y' \setminus \{e'\}))$$

and

$$\ell(\min(X')) = \ell(\min(Y')).$$

So

$$|\min(Y' \setminus \{e'\})| < |\min(Y')|$$

and

$$\min(Y' \setminus \{e'\}) \subsetneq \min(Y').$$

It follows that $e' \in \min(Y')$. Hence e' is concurrent with all events in $Y' \setminus Y$, and $Y \xrightarrow{A} Y'$, as required.

(2) $\min(X') = \min(X)$.
Let $M = \min(X)$, $N = \min(Y)$. By Lemma 5.4, we have $R(M, N)$.
Let configuration structures $\mathscr{C}_M$ and $\mathscr{D}_N$ be as in Definition 5.5. Let R' be the RHS-IB between $\mathscr{C}_M$ and $\mathscr{D}_N$ of Lemma 5.7.
We have $R'(X \setminus M, Y \setminus N)$ and $X \setminus M \xrightarrow{A}_{\mathscr{C}_M} X' \setminus M$. Clearly, $M \neq \emptyset$, since $|X'| > 0$.
So, by induction, there is Y' such that $Y \setminus N \xrightarrow{A}_{\mathscr{D}_N} Y' \setminus N$ with $R'(X' \setminus M, Y' \setminus N)$.
So $Y \xrightarrow{A}_{\mathscr{D}} Y'$ and $R(X', Y')$, as required. □

We can use the same method as Lemma 5.8 to show the following result.

Lemma 5.9. Let stable configuration structures $\mathscr{C}, \mathscr{D}$ be related by RHS-IB R. Suppose $R(X, Y)$ and $X \overset{A}{\rightsquigarrow}_{\mathscr{C}} X'$. Then there is Y' such that $Y \overset{A}{\rightsquigarrow}_{\mathscr{D}} Y'$ and $R(X', Y')$.

Proof. Much as in the proof of Lemma 5.8, we proceed by induction on $|X|$, with a very similar pair of cases.

For (1), note that we start by reversing from X in a single event transition using an element $e \in \min(X) \setminus \min(X')$, and we then do an $A \setminus \{a\}$ reverse step. If we did the $A \setminus \{a\}$ reverse step followed by the single a reverse transition, then on the $\mathscr{D}$ side we leave open the possibility that e' causes the remaining events of $Y' \cup \{e'\} \xrightarrow{A\setminus\{a\}} Y$.

For (2), we define R' in exactly the same way, and everything works much as before. □

Combining these results, we get the following theorem.

Theorem 5.10. On stable configuration structures, $\approx_{rhs\text{-}ib} = \approx_{rs\text{-}sb}$.

Proof. Clearly, any RS-SB is an RHS-IB. Also, any RHS-IB is an RS-SB, by Lemmas 5.8 and 5.9. □

5.2. *Reverse depth-preserving equivalences*

Recall that in the forward-only case, DB equivalence is strictly stronger than SB equivalence (Proposition 3.21). In this section we shall show that, by contrast, in the reverse case, the corresponding equivalences are equal (Theorem 5.24). The technical development will also be needed later when we give an improvement of Bednarczyk's result that, in the absence of auto-concurrency, RI-IB equivalence has the same power as H-H equivalence (Section 9).

We define reverse depth transitions and reverse equidepth transitions as follows.

Definition 5.11. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $A \in \mathbb{N}^{\mathsf{Act}}$, $k \in \mathbb{N}$. Then $X \overset{A,k}{\rightsquigarrow}_{\mathscr{C}} X'$ if and only if $X' \overset{A,k}{\rightarrow}_{\mathscr{C}} X$. Also $X \overset{A,=}{\rightsquigarrow}_{\mathscr{C}} X'$ if and only if $X' \overset{A,=}{\rightarrow}_{\mathscr{C}} X$.

This enables us to formulate equivalences $\approx_{rd\text{-}fb}$ and $\approx_{rd\text{-}h}$ in an analogous fashion to $\approx_{rs\text{-}fb}$, and so on. We also have equidepth and homogeneous equidepth versions of the reverse step equivalences, which we denote by $\approx_{res\text{-}fb}$ and $\approx_{rhes\text{-}fb}$. We will omit the definitions here as the pattern is clear.

We will now show that RS-SB equivalence can be characterised as RD-DB equivalence, and as RHES-IB equivalence.

Proposition 5.12. On stable configuration structures, we have:

(1) $\approx_{rd\text{-}ib} \subseteq \approx_{rs\text{-}ib} \subseteq \approx_{rhs\text{-}ib}$;
(2) $\approx_{rd\text{-}ib} \subseteq \approx_{res\text{-}ib} \subseteq \approx_{rhes\text{-}ib}$.

Proof. These are reverse counterparts to inclusions already shown for the corresponding forward-only equivalences in Propositions 3.21 and 3.28, and are shown in much the same way. □

We will show later (Proposition 5.26) that all the equivalences mentioned in Proposition 5.12 are in fact equal to each other. First we need some further results.

Lemma 5.13. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Let R be an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then $R(\min(X), \min(Y))$.

Proof. The proof is similar to the proof of Lemma 5.4. Note that all events in $\min(X)$ have depth one, and any transition involving only minimal elements is an equidepth one. □

Lemma 5.14. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Let R be an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. Let $M \in C_{\mathscr{C}}$ be such that $\min(M) = M$. Similarly, let $N \in C_{\mathscr{D}}$ be such that

$\min(N) = N$. Suppose also that $R(M, N)$. If we define $R_{M,N}$ by

$$R_{M,N} = \{(X \setminus M, Y \setminus N) : R(X, Y),\ \min(X) = M,\ \min(Y) = N\},$$

then $R_{M,N}$ is an RHES-IB between $\mathscr{C}_M$ and $\mathscr{D}_N$.

Proof. The proof is similar to the proof of Lemma 5.7, but using Lemma 5.13 instead of Lemma 5.4. □

Definition 5.15. Let $\mathscr{C}$ be a stable configuration structure. For $m, n \in \mathbb{N}$ ($m \leqslant n$) and $X \in C_{\mathscr{C}}$, let

$$X_{\leqslant n} \stackrel{\text{df}}{=} \{e \in X : \mathsf{depth}_X(e) \leqslant n\}$$
$$X_{\geqslant n} \stackrel{\text{df}}{=} \{e \in X : \mathsf{depth}_X(e) \geqslant n\}$$
$$X_{[m,n]} \stackrel{\text{df}}{=} \{e \in X : m \leqslant \mathsf{depth}_X(e) \leqslant n\}.$$

It is clear that $X_{\leqslant n}$ is a configuration since it is a left-closed subset of a configuration. Also, $X_{\leqslant 1} = \min(X)$. For large enough n, we have $X_{\leqslant n} = X$.

Proposition 5.16. Suppose R is an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $R(X_{\leqslant n}, Y_{\leqslant n})$.

Proof. Suppose $R(X, Y)$. We define configuration structures $\mathscr{C}_n, \mathscr{D}_n$ by

$$\mathscr{C}_0 \stackrel{\text{df}}{=} \mathscr{C}$$
$$\mathscr{C}_{n+1} \stackrel{\text{df}}{=} (\mathscr{C}_n)_{X_{[n+1,n+1]}}$$

(see Definition 5.5), and similarly for $\mathscr{D}_n$. Note that $X_{[n+1,n+1]} = \min(X_{\geqslant n+1})$. To ensure that $\mathscr{C}_n$ is well defined, we need to show that $X_{\geqslant n+1}$ (and hence $X_{[n+1,n+1]}$) is a configuration of $\mathscr{C}_n$, which can be done by induction on n (this is easy, so the details are omitted here). We can show similarly that $\mathscr{D}_n$ is well defined.

We also define RHES-IBs R_n as follows:

$$R_0 \stackrel{\text{df}}{=} R$$
$$R_{n+1} \stackrel{\text{df}}{=} (R_n)_{X_{[n+1,n+1]}, Y_{[n+1,n+1]}}$$

(see Lemma 5.14). To ensure that R_{n+1} is well defined (and an RHES-IB), we need $R_n(X_{[n+1,n+1]}, Y_{[n+1,n+1]})$ to hold. We can prove $R_n(X_{\geqslant n+1}, Y_{\geqslant n+1})$ by an easy induction, which we again omit here. Using Lemma 5.13, we then deduce

$$R_n(X_{[n+1,n+1]}, Y_{[n+1,n+1]}).$$

Now we show $R(X_{\leqslant n}, Y_{\leqslant n})$. If $n = 0$, this is just $R(\emptyset, \emptyset)$, which is true by definition. So we suppose $n \geqslant 1$ and show by induction on i that for $1 \leqslant i \leqslant n$, we have $R_{n-i}(X_{[n-i+1,n]}, Y_{[n-i+1,n]})$:

— $i = 1$:

For this case we need $R_{n-1}(X_{[n,n]}, Y_{[n,n]})$, which we have already shown.

— Suppose $R_{n-i}(X_{[n-i+1,n]}, Y_{[n-i+1,n]})$, with $i < n$.
Then

$$R_{n-i-1}(X_{[n-i+1,n]} \cup X_{[n-i,n-i]}, Y_{[n-i+1,n]} \cup Y_{[n-i,n-i]}),$$

that is,

$$R_{n-(i+1)}(X_{[n-(i+1)+1,n]}, Y_{[n-(i+1)+1,n]}),$$

as required.

Putting $i = n$, we get $R_0(X_{[1,n]}, Y_{[1,n]})$, that is, $R(X_{\leqslant n}, Y_{\leqslant n})$. □

Corollary 5.17. Suppose R is an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $\ell(X_{[n,n]}) = \ell(Y_{[n,n]})$.

Proof. The result follows from Proposition 5.16 and Lemma 5.3. □

Proposition 5.18. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Suppose R is an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. If $X \xrightarrow{a,k}_{\mathscr{C}} X'$, $Y \xrightarrow{a,k'}_{\mathscr{D}} Y'$, with $R(X, Y)$, $R(X', Y')$, then $k = k'$.

Proof. By Corollary 5.17, we know that X, Y (and X', Y') have the same multisets of events at each level n. Hence, the single events in $X' \setminus X$ and $Y' \setminus Y$ must have the same depth. □

Lemma 5.19. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures, and let R be an RHES-IB between $\mathscr{C}$ and $\mathscr{D}$. Then R is an RD-DB.

Proof. Suppose $R(X, Y)$ and $X \xrightarrow{a,k}_{\mathscr{C}} X'$. Since R is an IB, there are Y', k' such that $Y \xrightarrow{a,k'}_{\mathscr{D}} Y'$ and $R(X', Y')$. But then $k = k'$ by Proposition 5.18. So R is a DB. For the reverse transitions, suppose $R(X, Y)$ and $X \overset{a,k}{\rightsquigarrow}_{\mathscr{C}} X'$. Then $X \overset{\{a\},=}{\rightsquigarrow}_{\mathscr{C}} X'$. Since R is an RHES-IB, there is Y' such that $Y \overset{\{a\},=}{\rightsquigarrow}_{\mathscr{D}} Y'$ and $R(X', Y')$. Hence $Y \overset{a,k'}{\rightsquigarrow}_{\mathscr{D}} Y'$ for some k'. But then $k = k'$ by a further use of Proposition 5.18. Thus R is an RD-DB. □

So if R is an RHES-IB and $R(X, X')$, then X and X' have a similar structure in that for each depth n, they have the same multisets of labelled events at that depth. We can say that X and X' have similar amounts of concurrency (including auto-concurrency): X and X' have the same depth and the same 'width' at each level n. Of course, this does not imply the stronger statement that X and X' are isomorphic, since we are not claiming that X and X' have the same causal relationships between the levels.

Finally, we can state versions of Proposition 5.16, Corollary 5.17, Proposition 5.18 and Lemma 5.19 for RHS-IBs instead of RHES-IBs.

Proposition 5.20. Suppose R is an RHS-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $R(X_{\leqslant n}, Y_{\leqslant n})$.

Proof. The proof is similar to the proof of Proposition 5.16, but using Lemmas 5.4 and 5.7 instead of Lemmas 5.13 and 5.14. □

Corollary 5.21. Suppose R is an RHS-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $\ell(X_{[n,n]}) = \ell(Y_{[n,n]})$.

Proof. The result follows from Proposition 5.20 and Lemma 5.3. □

Proposition 5.22. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Suppose R is an RHS-IB between $\mathscr{C}$ and $\mathscr{D}$. If $X \overset{a,k}{\rightarrow}_{\mathscr{C}} X'$, $Y \overset{a,k'}{\rightarrow}_{\mathscr{D}} Y'$, with $R(X, Y)$, $R(X', Y')$, then $k = k'$.

Proof. The proof is similar to the proof of Proposition 5.18, but using Corollary 5.21 instead of Corollary 5.17. □

Lemma 5.23. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures, and let R be an RHS-IB between $\mathscr{C}$ and $\mathscr{D}$. Then R is an RD-DB.

Proof. The proof is similar to the proof of Lemma 5.19, but using Proposition 5.22 instead of Proposition 5.18. □

Theorem 5.24. On stable configuration structures, $\approx_{rd\text{-}db} = \approx_{rhes\text{-}ib} = \approx_{rs\text{-}sb}$.

Proof. Let R be an RD-DB between $\mathscr{C}$ and $\mathscr{D}$.

Then R is an RHES-IB and an RHS-IB by Proposition 5.12. So, by Theorem 5.10, R is an RS-SB.

Now suppose R is an RHES-IB. Then R is an RD-DB by Lemma 5.19.

Finally, we suppose R is an RS-SB. Then R is an RHS-IB, so R is an RD-DB by Lemma 5.23. □

5.3. *The reverse step hierarchy*

We have seen in Section 5.1 that reverse steps are more powerful than forward steps: $\approx_{rs-ib} \subsetneq \approx_{ri-sb}$. We have also seen that reverse depth observations are equivalent to reverse steps (Theorem 5.24). As a consequence, there is a considerable collapse in the hierarchy of equivalences in the 'reverse step family' compared with the reverse single-event equivalences.

Proposition 5.25. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. The following are equivalent:

(1) $\mathscr{C} \approx_{rs\text{-}ib} \mathscr{D}$
(2) $\mathscr{C} \approx_{rs\text{-}db} \mathscr{D}$
(3) $\mathscr{C} \approx_{rs\text{-}sb} \mathscr{D}$
(4) $\mathscr{C} \approx_{rs\text{-}hsb} \mathscr{D}$
(5) $\mathscr{C} \approx_{rs\text{-}esb} \mathscr{D}$
(6) $\mathscr{C} \approx_{rs\text{-}hesb} \mathscr{D}$.

Proof. The implications (2)⇒(3)⇒(4)⇒(1) and (2)⇒(5)⇒(6)⇒(1) hold by the corresponding results for the forward-only equivalences, as shown in Figure 7. Hence, it is enough to show (1)⇒(2).

Suppose R is an RS-IB. Then it is an RHS-IB. So it is an RD-DB by Lemma 5.23. Combining these results, R is an RS-DB, as required. □

Proposition 5.26. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. The following are equivalent:

(1) R is an RS-IB
(2) R is an RHS-IB
(3) R is an RD-IB
(4) R is an RES-IB
(5) R is an RHES-IB.

Proof. (1) ⇒ (2) is immediate from the definition.
(2) ⇒ (3) holds by Lemma 5.23.
(3) ⇒ (4) holds by (the proof of) Proposition 5.12.
(4) ⇒ (5) is immediate from the definition.
(5) ⇒ (3) holds by Lemma 5.19.
(3) ⇒ (1) holds by (the proof of) Proposition 5.12. □

Corollary 5.27. Let $\approx_{fb}$ be any of the forward-only equivalences in Figure 7. Then on stable configuration structures, we have

$$\approx_{rs\text{-}fb} = \approx_{rhs\text{-}fb} = \approx_{rd\text{-}fb} = \approx_{res\text{-}fb} = \approx_{rhes\text{-}fb}.$$

Proof. The result is immediate from Proposition 5.26. □

Proposition 5.28. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. Let R be an RI-WH bisimulation between $\mathscr{C}$ and $\mathscr{D}$. Then R is an RD-WH bisimulation.

Proof. Suppose $R(X, Y)$ and $X \overset{a,k}{\rightsquigarrow}_{\mathscr{C}} X'$. Then $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$, so there is Y' such that $Y \overset{a}{\rightsquigarrow}_{\mathscr{C}} Y'$ with $R(X', Y')$. Since R is a WH bisimulation, $X \cong Y$ and $X' \cong Y'$. Let $X \setminus X' = \{e\}$ and $Y \setminus Y' = \{e'\}$. Then $\mathsf{depth}_X(e) = \mathsf{depth}_Y(e')$ since depth is preserved by isomorphism. Hence $Y \overset{a,k}{\rightsquigarrow}_{\mathscr{C}} Y'$. Therefore R is an RD-WH, as required. □

Corollary 5.29. On stable configuration structures, we have

$$\approx_{rs\text{-}wh} = \approx_{ri\text{-}wh}$$
$$\approx_{rs\text{-}whpb} = \approx_{ri\text{-}whpb}$$
$$\approx_{rs\text{-}h} = \approx_{ri\text{-}h}.$$

Proof. The result is immediate from Proposition 5.28 and Corollary 5.27. □

Figure 18 shows the reverse single-event or step equivalences arranged in a single ordering. Recall that the arrows with open heads represent inclusions that are not known to be proper, and the arrows with solid heads represent proper inclusions.

Equivalences $\approx_{rs\text{-}sb}$ and $\approx_{rs\text{-}db}$ are the same as $\approx_{rs\text{-}ib}$ by Proposition 5.25, and the remaining equations hold by Corollary 5.29. The various inclusions follow from other results in this section. These were all established in Section 4, with two exceptions:

— $\approx_{rs\text{-}pb} \subsetneq \approx_{ri\text{-}pb}$:
For this, we can use Example 4.6, where the two structures are RI-PB equivalent, but not DB equivalent. Hence they are not RI-DB equivalent. But from Figure 18, we see that they cannot then be RS-PB equivalent.

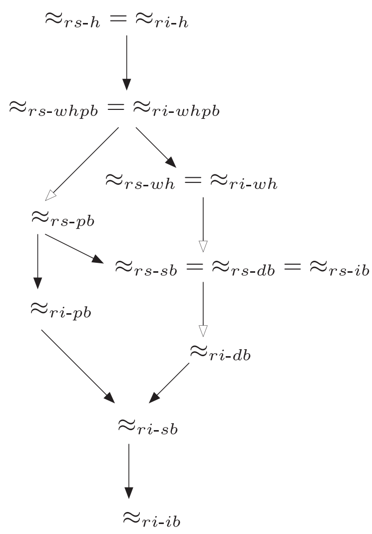

Fig. 18. Equivalences with reverse single-event or step transitions.

— $\approx_{rs\text{-}pb} \subsetneq \approx_{rs\text{-}sb}$:
For this, we can use Example 4.7, where the two structures are RI-WH equivalent, but not PB equivalent. They are therefore RS-WH, and thus RS-SB, equivalent, but not RS-PB equivalent.

We conclude by noting that extending the allowed observations to include reverse step transitions increases the distinguishing power of the coarser equivalences in Figure 8, namely $\approx_{rs\text{-}ib} \subsetneq \approx_{ri\text{-}ib}$, $\approx_{rs\text{-}sb} \subsetneq \approx_{ri\text{-}sb}$ and $\approx_{rs\text{-}pb} \subsetneq \approx_{ri\text{-}pb}$ in Figure 18. However, it has no effect on the three finer history-preserving-like equivalences $\approx_{ri\text{-}wh}$, $\approx_{ri\text{-}whpb}$ and $\approx_{ri\text{-}h}$ since we have $\approx_{rs\text{-}wh} = \approx_{ri\text{-}wh}$, $\approx_{rs\text{-}whpb} = \approx_{ri\text{-}whpb}$ and $\approx_{rs\text{-}wh} = \approx_{ri\text{-}wh}$.

6. Reverse pomset transitions

The reverse pomset transition relation, represented by a wavy arrow, is defined in a corresponding way to the single event reverse transition, namely, by simply inverting the standard forward pomset version of Definition 3.6.

Definition 6.1. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and u be a pomset as in Definition 3.5. Then $X \overset{u}{\rightsquigarrow}_{\mathscr{C}} X'$ if and only if $X' \xrightarrow{u}_{\mathscr{C}} X$.

In much the same way as for single reverse event transitions, we can now define reverse pomset versions $\approx_{rp\text{-}fb}$ and $\approx_{rp\text{-}h}$ of all the equivalences defined in Section 3 – the definitions are similar to Definitions 4.2 and 4.3, so we omit them here. We can also define equivalences $\approx_{rhs\text{-}fb}$ in a corresponding way.

Note that any of these bisimulations gives us an order isomorphism. Assume that R is an RP-FB and let $R(X, Y)$. If we reverse X fully, namely, $X \overset{u}{\rightsquigarrow}_{\mathscr{C}} \emptyset$ with $u = [(X, <_X, \ell_{\mathscr{C}} \upharpoonright X)]_{\cong}$, then this must be matched by $Y \overset{u}{\rightsquigarrow}_{\mathscr{D}} \emptyset$ by the definition of R. Thus, u is the

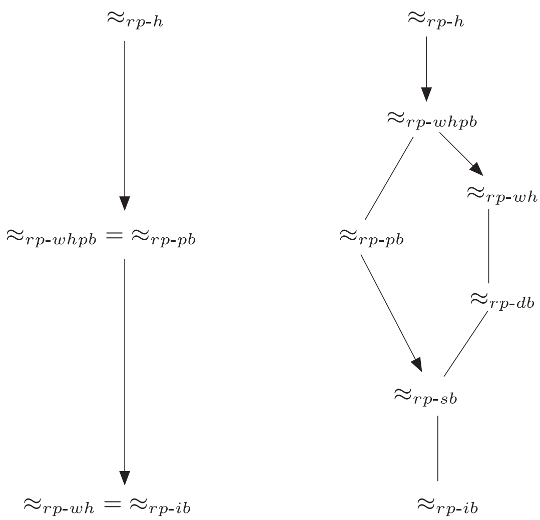

Fig. 19. Equivalences with reverse pomset transitions.

pomset associated with Y and $X \cong Y$. As a consequence, an RP-IB is also an RP-WH and an RP-PB is also an RP-WHPB.

Theorem 6.2. On stable configuration structures, $\approx_{rp\text{-}ib} = \approx_{rp\text{-}sb} = \approx_{rp\text{-}db} = \approx_{rp\text{-}wh}$, and $\approx_{rp\text{-}pb} = \approx_{rp\text{-}whpb}$.

Example 4.7 holds for $\approx_{rp\text{-}sb}$, but not for $\approx_{rp\text{-}pb}$. It also holds for $\approx_{rp\text{-}wh}$, but not for $\approx_{rp\text{-}whpb}$. Finally, Example 4.8 shows that $\approx_{rp\text{-}h}$ is strictly contained in $\approx_{rp\text{-}whpb}$.

Figure 19 shows the full hierarchy of reverse pomset equivalences on the right. Plain lines (as opposed to arrows) indicate coincidence of relations. Note that there are only three distinct relations, so when we take conceptually simplest relations as the representatives of the sets of coinciding bisimulations, we obtain the left hierarchy in Figure 19. It is an open question as to whether the reverse pomset equivalences strictly imply the corresponding reverse step equivalences (for example, whether $\approx_{rp\text{-}h} \subsetneq \approx_{rs\text{-}h}$).

7. Hereditary equivalences

The reverse versions of history preserving bisimulation, weak history bisimulation and weak history pomset bisimulation are defined by additionally insisting that the related configurations also have matching reverse behaviour (reverse single events, reverse steps or reverse pomsets). The hereditary property states that if there is an order isomorphism between two configurations, this isomorphism is preserved under reverse single event transitions. Therefore, as in the definition of history-preserving bisimulation (Definition 3.10), it is useful to name the isomorphism explicitly when defining WH and WHPB. We restate these definitions here for clarity.

Definition 7.1. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}} \times \mathscr{P}(E_{\mathscr{C}} \times E_{\mathscr{D}})$ is a *weak history-preserving* (*WH*) *bisimulation* between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset, \emptyset) \in R$ and if $(X, Y, f) \in R$ and $a \in \mathsf{Act}$, then:

— f is an isomorphism between $(X, <_X, \ell_{\mathscr{C}} \upharpoonright X)$ and $(Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$;
— if $X \xrightarrow{a}_{\mathscr{C}} X'$, then $\exists Y', f'.\ Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $(X', Y', f') \in R$;
— if $Y \xrightarrow{a}_{\mathscr{D}} Y'$, then $\exists X', f'.\ X \xrightarrow{a}_{\mathscr{C}} X'$ and $(X', Y', f') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are WH equivalent ($\mathscr{C} \approx_{wh} \mathscr{D}$) if and only if there is a WH bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

Definition 7.2. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures. A relation $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}} \times \mathscr{P}(E_{\mathscr{C}} \times E_{\mathscr{D}})$ is a *weak history-preserving pomset bisimulation* (*WHPB*) between $\mathscr{C}$ and $\mathscr{D}$ if $(\emptyset, \emptyset, \emptyset) \in R$ and if $(X, Y, f) \in R$ and u is a pomset over Act, then:

— f is an isomorphism between $(X, <_X, \ell_{\mathscr{C}} \upharpoonright X)$ and $(Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$;
— if $X \xrightarrow{u}_{\mathscr{C}} X'$, then $\exists Y', f'.\ Y \xrightarrow{u}_{\mathscr{D}} Y'$ and $(X', Y', f') \in R$;
— if $Y \xrightarrow{u}_{\mathscr{D}} Y'$, then $\exists X', f'.\ X \xrightarrow{u}_{\mathscr{C}} X'$ and $(X', Y', f') \in R$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are WHPB equivalent ($\mathscr{C} \approx_{whpb} \mathscr{D}$) if and only if there is a WHPB between $\mathscr{C}$ and $\mathscr{D}$.

We can now define hereditary versions of the three forward equivalences that use history isomorphisms. Let FH stand for any of WH, WHPB and H, and let $\approx_{fh}$ be the associated equivalence.

Definition 7.3. Let $\mathscr{C} = (C, \ell)$ be a stable configuration structure and let $a \in \mathsf{Act}$. Then $R \subseteq C_{\mathscr{C}} \times C_{\mathscr{D}} \times \mathscr{P}(E_{\mathscr{C}} \times E_{\mathscr{D}})$ is an H-FH if and only if R is an FH and if $R(X, Y, f)$, then for any $a \in \mathsf{Act}$:

— if $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$, then $\exists Y', f'.\ Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$, $(X', Y', f') \in R$ and $f \upharpoonright X' = f'$;
— if $Y \overset{a}{\rightsquigarrow}_{\mathscr{D}} Y'$, then $\exists X', f'.\ X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$, $(X', Y', f') \in R$ and $f \upharpoonright X' = f'$.

We say that $\mathscr{C}$ and $\mathscr{D}$ are H-FH equivalent ($\mathscr{C} \approx_{h\text{-}fh} \mathscr{D}$) if and only if there is an H-FH bisimulation between $\mathscr{C}$ and $\mathscr{D}$.

The hereditary property was introduced in Bednarczyk (1991), and van Glabbeek and Goltz (2001) showed that the property is invariant under replacement of the single event transitions by step or pomset transitions when used in the definition of H-H bisimulation.

It is clear from the definitions of H-H, H-WHPB and H-WH that $\approx_{h\text{-}h} \subseteq \approx_{h\text{-}whpb} \subseteq \approx_{h\text{-}wh}$. Example 4.7 shows that $\approx_{h\text{-}whpb}$ is strictly contained in $\approx_{h\text{-}wh}$, and Example 4.8 shows that $\approx_{h\text{-}h}$ is strictly contained in $\approx_{h\text{-}whpb}$.

Figure 20 gives the full hierarchy of hereditary equivalences and reverse pomset equivalences (on the right). An open question remains if the hereditary equivalences are strictly finer than the corresponding reverse pomset equivalences.

8. Refinement

Van Glabbeek and Goltz (2001) investigated the refinement of actions, and showed that the finest two bisimulation equivalences they investigated, namely $\approx_h$ and $\approx_{h\text{-}h}$, are

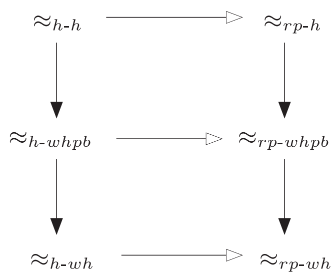

Fig. 20. Hereditary equivalences and equivalences with reverse pomset transitions.

preserved under refinement, whereas the other bisimulation equivalences they considered, namely $\approx_{ib}$, $\approx_{sb}$, $\approx_{pb}$, $\approx_{wh}$ and $\approx_{whpb}$ are not. We can strengthen their negative results by showing that any equivalence lying between $\approx_{h\text{-}whpb}$ and $\approx_{ib}$ is not preserved under refinement. See van Glabbeek and Goltz (2001, Definition 5.4) for the definition of a refinement function *ref* and the refinement $ref(\mathscr{C})$ of a stable configuration structure $\mathscr{C}$.

Proposition 8.1. Let $\approx_*$ be such that $\approx_{h\text{-}whpb} \subseteq \approx_* \subseteq \approx_{ib}$. Then $\approx_*$ is not preserved by refinement.

Proof. Consider the structures $\mathscr{E}$, $\mathscr{F}$ of Example 4.8. We have $\mathscr{E} \approx_{h\text{-}whpb} \mathscr{F}$. Hence $\mathscr{E} \approx_* \mathscr{F}$. If we refine a to the structure $a' \to a''$ (that is, $ref(a) = a' \to a''$), then in $ref(\mathscr{E})$, any sequence of transitions $\xrightarrow{a'}\xrightarrow{a''}\xrightarrow{a'}$ leaves $\xrightarrow{b}$ possible. However, in $ref(\mathscr{F})$, if we perform $\xrightarrow{a'_3}\xrightarrow{a''_3}\xrightarrow{a'_2}$, then $\xrightarrow{b}$ is impossible – see Figure 21. Thus, $ref(\mathscr{E}) \not\approx_{ib} ref(\mathscr{F})$, so $ref(\mathscr{E}) \not\approx_* ref(\mathscr{F})$. □

Whether the equivalences lying between $\approx_h$ and $\approx_{h\text{-}h}$, namely, $\approx_{ri\text{-}h}$ and $\approx_{rp\text{-}h}$, are preserved by refinement remains an open question. As we have seen, the inclusions $\approx_{h\text{-}h} \subseteq \approx_{rp\text{-}h} \subseteq \approx_{ri\text{-}h}$ may or may not be strict.

9. Excluding equidepth auto-concurrency

In this section we extend a result of Bednarczyk stating that RI-IB equivalence is as strong as H-H equivalence in the absence of auto-concurrency. We make use of results from Section 5.2 on depth-respecting bisimulations, and show that Bednarczyk's result still holds under the weaker assumption of no *equidepth* auto-concurrency. We conclude the section by showing the complete (and considerably simplified) hierarchy of equivalences under this assumption.

Definition 9.1. We say a stable configuration structure $\mathscr{C}$ is *without auto-concurrency* if for any $X \in C_{\mathscr{C}}$ and any $d, e \in X$, if $d\ co_X\ e$ and $\ell(d) = \ell(e)$, then $d = e$.

Bednarczyk (1991) showed the following result.

Theorem 9.2. For prime event structures, in the absence of auto-concurrency, $\approx_{ri\text{-}ib} = \approx_{h\text{-}h}$.

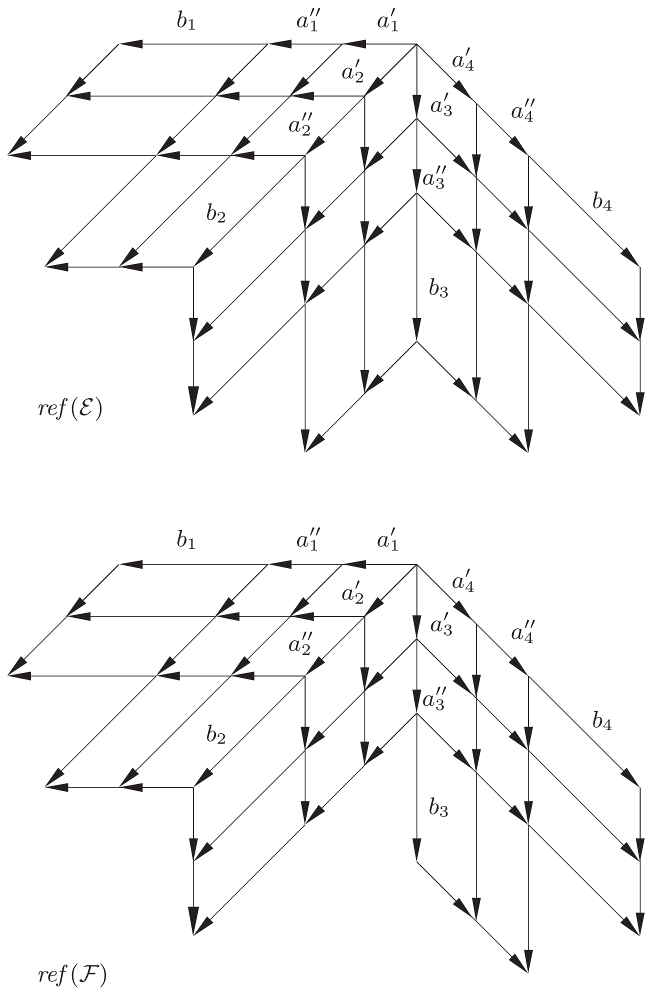

Fig. 21. Transition diagrams for $ref(\mathscr{E})$ and $ref(\mathscr{F})$.

Before extending Theorem 9.2 to give Theorem 9.7, we need a definition. We say that a configuration structure is *without equidepth auto-concurrency* if for any pair of distinct concurrent events, they cannot have both the same label and the same depth. This is less restrictive than 'without auto-concurrency' in that we allow auto-concurrent events, providing they are at different depths. For example, the configuration structure $a \mid (b.a)$ exhibits auto-concurrency between the two events labelled a, but not equidepth auto-concurrency, since the two a events are at depths one and two, respectively.

Definition 9.3. We say that a stable configuration structure $\mathscr{C}$ is *without equidepth auto-concurrency* if for any $X \in C_{\mathscr{C}}$ and any $d, e \in X$, if $d \; co_X \; e$ with both $\ell(d) = \ell(e)$ and $\mathsf{depth}_X(d) = \mathsf{depth}_X(e)$, then $d = e$.

Note that without equidepth auto-concurrency, an RI-IB is an RHES-IB since all reverse homogeneous equidepth steps must have just a single event: $X \overset{A,=}{\rightsquigarrow_{\mathscr{C}}} X'$ with A homogeneous implies that $|A| = 1$. Therefore, we immediately have the following versions of Proposition 5.16, Corollary 5.17 and Proposition 5.18, where $X_{\leqslant n}$ and $X_{[n,n]}$ are as in Definition 5.15.

Proposition 9.4. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures without equidepth auto-concurrency and suppose R is an RI-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $R(X_{\leqslant n}, Y_{\leqslant n})$.

Corollary 9.5. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures without equidepth auto-concurrency and suppose R is an RI-IB between $\mathscr{C}$ and $\mathscr{D}$. If $R(X, Y)$, then for each $n \in \mathbb{N}$, we have $\ell(X_{[n,n]}) = \ell(Y_{[n,n]})$.

Note that the no equidepth auto-concurrency condition means that $\ell(X_{[n,n]})$, $\ell(Y_{[n,n]})$ are sets rather than multisets since events at the same depth must be concurrent.

Proposition 9.6. Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures without equidepth auto-concurrency and suppose R is an RI-IB between $\mathscr{C}$ and $\mathscr{D}$. If $X \overset{a,k}{\rightarrow_{\mathscr{C}}} X'$, $Y \overset{a,k'}{\rightarrow_{\mathscr{D}}} Y'$, with $R(X, Y)$, $R(X', Y')$, then $k = k'$.

Proposition 9.6 may not hold in the presence of equidepth auto-concurrency. For example, we have $a\,|\,a \approx_{ri\text{-}ib} a.a$, but the two a events are both at depth one in the case of $a\,|\,a$, whereas they are at depths one and two, respectively, in the case of $a.a$.

Theorem 9.7. In the absence of equidepth auto-concurrency, $\approx_{ri\text{-}ib} = \approx_{h\text{-}h}$.

Proof. The proof is rather different from that of Theorem 9.2, and, with the help of Corollary 9.5 and Proposition 9.6, is quite short.

Let $\mathscr{C}, \mathscr{D}$ be stable configuration structures without equidepth auto-concurrency and assume R is an RI-IB between $\mathscr{C}$ and $\mathscr{D}$.

Suppose $R(X, Y)$ and define $f_{X,Y} : X \to Y$ by $f_{X,Y}(d) = e$ where e is the unique $e \in Y$ such that $\mathsf{depth}_Y(e) = \mathsf{depth}_X(d)$ and $\ell(e) = \ell(d)$. Then $f_{X,Y}$ is well defined and is a bijection by Corollary 9.5. Clearly, f preserves labels and depth.

We next show $f_{X,Y}$ is order preserving. Note that events in X and Y are determined uniquely by their depth and label. Suppose $d <_X d'$, with $\ell(d) = a$, $\ell(d') = a'$, $\mathsf{depth}_X(d) = k$ and $\mathsf{depth}_X(d') = k'$. In order to show a contradiction, we suppose $f_{X,Y}(d) \not<_Y f_{X,Y}(d')$. Then there is $Y'' \in C_{\mathscr{D}}$ such that $f_{X,Y}(d') \in Y'' \subsetneq Y$ and $f_{X,Y}(d) \notin Y''$. So there is a series of reverse transitions starting from Y in which we reverse $f_{X,Y}(d)$ but not $f_{X,Y}(d')$, that is,

$$Y \overset{a_1,k_1}{\rightsquigarrow_{\mathscr{D}}} Y_1 \cdots \overset{a_n,k_n}{\rightsquigarrow_{\mathscr{D}}} Y_n \overset{a,k}{\rightsquigarrow_{\mathscr{D}}} Y' \supseteq Y'',$$

with $\ell(a_i) \neq a'$ or $\mathsf{depth}_Y(a_i) \neq k'$ for $i = 1, \ldots, n$.

Since R is an RI-IB, there is a corresponding series of reverse transitions starting from X. By Proposition 9.6, the corresponding events must have the same depth. So

$$X \overset{a_1,k_1}{\rightsquigarrow_{\mathscr{C}}} X_1 \cdots \overset{a_n,k_n}{\rightsquigarrow_{\mathscr{C}}} X_n \overset{a,k}{\rightsquigarrow_{\mathscr{C}}} X',$$

with $R(X_i, Y_i)$ for $i = 1, \ldots, n$ and $R(X', Y')$.

But then the last transition $X_n \overset{a,k}{\rightsquigarrow}_{\mathscr{C}} X'$ must have d as its underlying event, and none of the previous n transitions can have d' as underlying event since either the depth or label does not match. This means that $d \not<_X d'$, which is a contradiction.

We can show symmetrically that $e <_Y e'$ implies $f_{X,Y}^{-1}(e) <_X f_{X,Y}^{-1}(e)$ (the definition of $f_{X,Y}^{-1}$ is just the same as that of $f_{X,Y}$ with X and Y swapped). Hence, $f_{X,Y}$ is order preserving.

Claim 1. If $X \xrightarrow{a}_{\mathscr{C}} X'$, $Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $R(X,Y)$, $R(X',Y')$, then $f_{X',Y'} \upharpoonright X = f_{X,Y}$.

Proof of Claim. Let $X' \setminus X = \{d\}$, $Y' \setminus Y = \{e\}$. We first show $f_{X',Y'}(d) = e$. By Proposition 9.6, there is k such that $X \xrightarrow{a,k}_{\mathscr{C}} X'$ and $Y \xrightarrow{a,k}_{\mathscr{D}} Y'$. So $\mathsf{depth}_{Y'}(e) = \mathsf{depth}_{X'}(d)$ and $\ell(e) = \ell(d)$. Hence, $f_{X',Y'}(d) = e$ by the definition of $f_{X',Y'}$. Take any $d' \in X$. Then $f_{X',Y'}(d') = e'$ where e' is the unique $e' \in Y'$ such that $\mathsf{depth}_{Y'}(e') = \mathsf{depth}_{X'}(d')$ and $\ell(e') = \ell(d')$. Since $f_{X',Y'}(d) = e$, we must have $e' \in Y$. Hence $f_{X',Y'}(d') = f_{X,Y}(d')$, and the Claim is shown. □

To complete the proof of the theorem, we now define $R'(X,Y,f)$ if and only if $R(X,Y)$ and $f = f_{X,Y}$ (any X,Y). We claim that R' is an H-H bisimulation between $\mathscr{C}$ and $\mathscr{D}$. We have shown that $f_{X,Y}$ is an isomorphism between $(X, <_X, \ell_{\mathscr{C}} \upharpoonright X)$ and $(Y, <_Y, \ell_{\mathscr{D}} \upharpoonright Y)$. Clearly, $R'(\emptyset, \emptyset, f_{\emptyset,\emptyset})$, so we now assume $R'(X,Y,f)$.

Suppose $X \xrightarrow{a}_{\mathscr{C}} X'$. Then there is Y' such that $Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $R(X',Y')$. So we obtain $R'(X',Y',f_{X',Y'})$. We also have $f_{X',Y'} \upharpoonright X = f_{X,Y}$ by the Claim.

The remaining cases where we suppose $Y \xrightarrow{a}_{\mathscr{D}} Y'$ and $X \overset{a}{\rightsquigarrow}_{\mathscr{C}} X'$ are handled similarly, again using the Claim.

This completes the proof of the theorem. □

We have strengthened Theorem 9.2 in two ways: we have used stable configuration structures rather than prime event structures and we have weakened the assumption of no auto-concurrency to no equidepth auto-concurrency.

Figure 22 shows the complete hierarchy of equivalences under the condition of no equidepth auto-concurrency. Since all the reverse equivalences are intermediate between $\approx_{h\text{-}h}$ and $\approx_{ri-ib}$, they all collapse to $\approx_{h\text{-}h} = \approx_{ri\text{-}ib}$ by Theorem 9.7.

Van Glabbeek and Goltz (2001) showed that in the absence of auto-concurrency, $\approx_h = \approx_{wh}$ (Theorem 9.2). It is not hard to check that their proof still works in the absence of equidepth auto-concurrency. Of course, $\approx_{whpb}$ is then also the same as $\approx_h$. Finally, as we have already observed, $\approx_{hesb} = \approx_{ib}$.

The inclusions in Figure 22 follow from those in Figure 7. All the arrows represent strict inclusions, and no other inclusions hold. We will now provide two examples to show that this is the case – we have to be careful not to use examples with equidepth auto-concurrency.

Example 9.8. In this example we modify the processes $\mathscr{E}$ and $\mathscr{F}$ of Example 3.30 by adding a new event labelled with c in parallel to form new processes $\mathscr{E}''$ and $\mathscr{F}''$ – see Figure 23. Then $\mathscr{E}'' \approx_{pb} \mathscr{F}''$ but $\mathscr{E}'' \not\approx_{esb} \mathscr{F}''$.

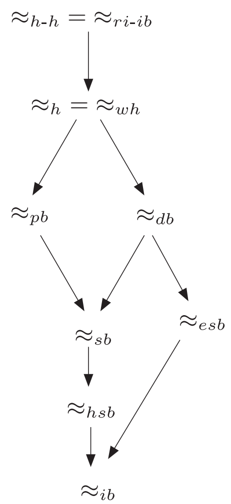

Fig. 22. The hierarchy with no equidepth auto-concurrency.

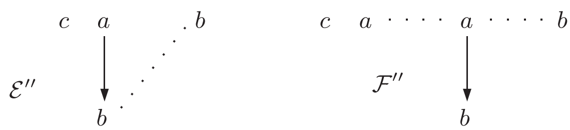

Fig. 23. Example 9.8.

Example 9.9. $a\,|\,(b.b) = (a\,|\,(b.b)) + b.(a\,|\,b)$ (where $a \neq b$) holds for $\approx_{sb}$ but not for $\approx_{pb}$ or $\approx_{esb}$.

We can complete the task of showing that all arrows represent strict inclusions, and that no other inclusions hold, using existing examples, as follows:

— Example 3.27 holds for $\approx_{esb}$ but not for $\approx_{hsb}$.
— Example 3.26 holds for $\approx_{hsb}$ but not for $\approx_{sb}$ or $\approx_{esb}$.
— Example 3.22 holds for $\approx_{db}$ but not for $\approx_{pb}$.
— Example 4.9 holds for $\approx_h$ but not for $\approx_{h\text{-}h} = \approx_{ri\text{-}ib}$.

10. Concluding remarks

We have investigated several forms of observations of reversible behaviour and have defined a wide range of reverse bisimulations by combining both forward and reverse observations. We have considered reverse single-event transitions, reverse steps, reverse depth-preserving transitions, reverse pomsets and the hereditary property of reverse behaviour.

Figure 24 shows relationships between the main forward-only equivalences and the range of reverse versions of these equivalences. Solid-headed arrows represent proper inclusions, open-headed arrows represent an inclusion not known to be proper and plain lines represent coincidence of equivalences. Recall that the absorption law (Example 4.9) separates the forward-only hierarchy from all the reverse bisimulations, and Example 4.8

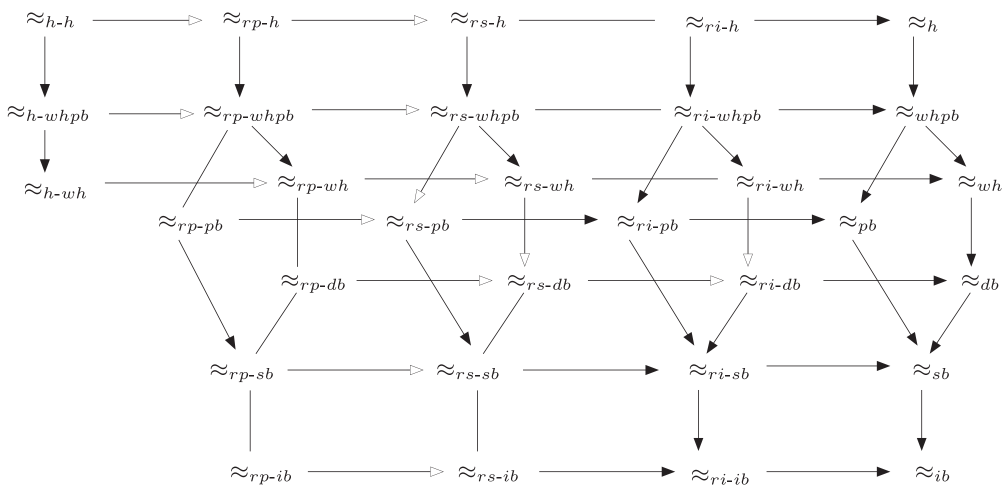

Fig. 24. The hierarchy of forward and reverse bisimulations.

separates the top row equivalences, namely those between $\approx_h$ and $\approx_{h\text{-}h}$, from all the remaining equivalences. Example 4.7 distinguishes $\approx_{whpb}$ from $\approx_{wh}$ and distinguishes all the *ri*-, *rs*-, *rp*- and *h*- versions of the two equivalences. Moreover, it separates $\approx_{pb}$ and $\approx_{sb}$ and all the *ri*-, *rs*-, *rp*- and *h*- versions of these bisimulations. There are still open questions about whether the equivalences related by open-headed arrows in Figure 24 are distinct. We conjecture that they are indeed distinct, and that they can be distinguished by examples yet to be found, and which may well be more complicated than those given in this paper.

We have strengthened Bednarczyk's result that, in the absence of auto-concurrency, RI-IB is as strong as H-H bisimulation (Bednarczyk 1991) by showing that we need only exclude events with equidepth auto-concurrency (Theorem 9.7). In Figure 22 we have given the full hierarchy of forward and reverse bisimulations for configuration structures with no equidepth auto-concurrency.

Bednarczyk also asked whether RS-SB was as strong as H-H. We can now answer his question in the negative as can be seen in Example 4.7. Moreover, we observe that there is a proper hierarchy in Figure 24 of reverse bisimulations that places RP-PB between H-H and RS-SB, namely, $\approx_{h\text{-}h} \subsetneq \approx_{rp\text{-}pb} \subsetneq \approx_{rs\text{-}sb} \subsetneq \approx_{ri\text{-}ib}$ (see Figure 25). The strongest reverse equivalence short of full H-H is H-WHPB, which has full reverse power. Example 4.8 shows that it is strictly less powerful than H-H. Thus, in some sense, we need a 'forward' component as well as a 'reverse' one.

We have shown in Proposition 5.25 that all the power of RS-SB equivalence resides in the reverse step transitions, with the forward steps being dispensable (that is, can be replaced with single-event transitions), namely, RS-SB and RS-IB coincide. In fact, the reverse steps can be restricted to those that are homogeneous (Theorem 5.10). One can even restrict attention to reverse homogeneous equidepth steps, where all events have the same depth (Theorem 5.24).

We have also proved that RS-SB equivalence preserves depth in the sense that corresponding events must have the same depth. We have introduced the notion of

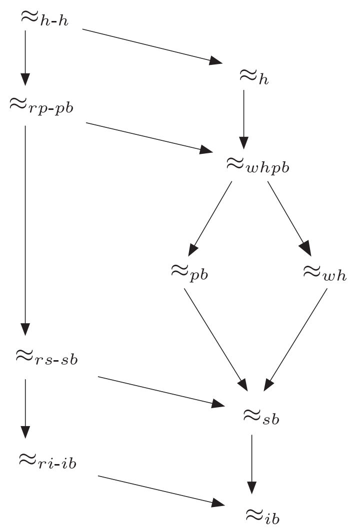

Fig. 25. The hierarchy of main bisimulations.

depth-respecting bisimulation (DB). In the forward-only direction, it turns out that DB is strictly stronger than SB. However, we have proved that reverse depth-respecting DB (RD-DB) has the same power as RS-SB (Theorem 5.24).

We have also investigated which of our reverse-type equivalences are preserved by refinement of actions. Van Glabbeek and Goltz showed that history-preserving bisimulation and hereditary history-preserving bisimulation are preserved by refinement. We have extended their study by showing (Proposition 8.1) that all the reverse-type equivalences considered here fail to be preserved by refinement, with the possible exception of the few sandwiched between H and H-H.

The full hierarchy of equivalences can be distilled into a hierarchy of important equivalences presented in Figure 25. All the main forward-only equivalences studied in the literature are present, as well as the four major reverse bisimulations studied here. Three of the reverse bisimulations (single-event transitions (RI-IB), step transitions (RS-SB) and pomset transitions (RP-PB)) are based on observations; by contrast H-H is based on the hereditary history-preserving property, which we do not consider to be observable. All the inclusions presented in Figure 25 are proper, and there are no other inclusions, with one possible exception arising from the fact that it is not known in general if an RS-SB is a WH. If one is interested in reverse bisimulations defined in terms of observations, then all equivalences in Figure 25 are suitable candidates except H and H-H. By contrast, if one is interested in an equivalence that is preserved by action refinement, then only H and H-H are useful (Proposition 8.1).

Figure 26 lists all of the examples used in the paper, and shows which equivalences are valid or invalid for them – we list the relevant strongest equivalences in the Valid column and the weakest equivalences in the Invalid column, with the convention that parentheses mean that the equivalence is implied by another equivalence, and, to save clutter, we write, for example, 'db' instead of $\approx_{db}$.

Example	Valid	Invalid
Example 3.12	ib, ri-ib	hsb, (sb), hesb
Example 3.13	sb, ri-sb, ri-esb	pb, db, rs-ib
Example 3.14	pb	db, ri-ib
Example 3.15	wh	pb, ri-ib
Example 3.16	whpb	h, ri-ib
Example 3.22	db	pb, wh, ri-ib
Example 3.26	hsb, hesb	sb, esb, ri-ib
Example 3.27	esb	hsb, ri-ib
Example 3.29	sb, ri-ib	hesb, pb, ri-sb
Example 3.30 $(\mathcal{E}, \mathcal{F})$	pb, esb	db, ri-ib
Example 3.30 $(\mathcal{E}', \mathcal{F}')$	pb, ri-pb	hesb
Example 4.5	ri-pb, h	ri-db
Example 4.6	ri-pb	db
Example 4.7	rp-sb, h-wh	pb
Example 4.8	h-whpb	h
Example 4.9	h	ri-ib
Figure 17	ri-hesb	esb, (db), (ri-esb)
Example 9.8	pb	esb
Example 9.9	sb, hesb	pb, esb, ri-ib

Fig. 26. Examples

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