# Program Analysis (70020)

## Control Flow Analysis

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- Not so trivial for more general languages e.g imperative languages with procedures as parameters, functional languages or object-oriented languages.
- A special analysis is required: Control Flow Analysis

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# Syntax of `Fun`

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e & ::= & t^{\ell} \\
t & ::= & c \mid x \mid \texttt{fn}\ x \ \texttt{=>}\ e_0 \mid e_1\ e_2 \\
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$$((\texttt{fn}\ x \texttt{ => } x^1)^2\ (\texttt{fn}\ y \texttt{ => } y^3)^4)^5$$

# An Example

```
let  f = fn x => x 1;
     g = fn y => y + 2;
     h = fn z => z + 3
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# Evaluating `Fun`

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- $\mathrm{eval}(\rho, e) = v$ can also be read as an specification for building an interpreter for the `Fun` language.
- We will use this specification just as a aid to help us understand the Control Flow Analysis.

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$$\textit{where}\ \ v = \begin{cases} \text{eval}(\rho, t_1^{\ell_1}) \text{ for } \text{eval}(\rho, t_0^{\ell_0}) = \texttt{true} \\ \text{eval}(\rho, t_2^{\ell_2}) \text{ for } \text{eval}(\rho, t_0^{\ell_0}) = \texttt{false} \end{cases}$$

# Environment Rules II [Provided in Exam]

$$\text{eval}(\rho, (\texttt{fn } x \texttt{ => } e_0)^\ell) = [(\texttt{fn } x \texttt{ => } e_0), \rho] \quad \text{closure creation}$$

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$$\text{eval}(\rho, (t_1^{\ell_1}\ t_2^{\ell_2})^{\ell}) = \text{eval}(\rho_0[x \mapsto v_2], e_0)$$ function application

*where* $\text{eval}(\rho, t_1^{\ell_1}) = [(\texttt{fn}\ x\ \texttt{=>}\ e_0), \rho_0] \wedge$
$\text{eval}(\rho, t_2^{\ell_2}) = v_2$

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In our imperative setting WHILE we might also allow variables to point to programs, e.g. $\ldots \mid [p := S]^{\ell} \mid p \mid \ldots$ Then, e.g.

$$\textbf{if}\ b\ \textbf{then}\ [p := S_1]^1\ \textbf{else}\ [p := S_2]^2;\ p$$

leads to the the question whether $(1, init(S_1))$ and/or $(1, init(S_2))$ should be in the control *flow*.

# CFA and Functional Programs

Consider the following `Fun` program:

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\begin{array}{ll}
\texttt{let} & f = \texttt{fn}\ x \texttt{=>}\ x\ 1\texttt{;} \\
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The aim of Control Flow Analysis is:

*For each function application, which functions may be applied*

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  - Abstract Domains and Specification
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- Context-Sensitive Analysis Concepts

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We will define a 0-CFA Analysis; the presentation requires two components:

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The result of a 0-CFA analysis is a pair $(\widehat{\mathsf{C}}, \widehat{\rho})$ where:

- $\widehat{\mathsf{C}}$ is the abstract cache associating abstract values with each labelled program point.
- $\widehat{\rho}$ is the abstract environment associating abstract values with each variable.

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Compare this with the Concrete Domain (see before):

$$
\begin{array}{llcll}
\rho & \in & \textbf{Env} & = & \textbf{Var} \to \textbf{Val} & \text{environments} \\
v & \in & \textbf{Val} & = & \mathbf{Z} \cup \textbf{Closure} & \text{values} \\
 & & \textbf{Closure} & ::= & [\texttt{fn}\ x \texttt{=>}\ e_0, \rho] & \text{closures}
\end{array}
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# Acceptable CFA

For the formulation of the 0-CFA analysis we shall write

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models e$$

for when $(\widehat{\mathsf{C}}, \widehat{\rho})$ is an acceptable Control Flow Analysis of the expression $e$. Thus the relation "$\models$" has functionality

$$\models : (\widehat{\textbf{Cache}} \times \widehat{\textbf{Env}} \times \textbf{Exp}) \rightarrow \{\texttt{true}, \texttt{false}\}$$

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Our Goal therefore is:

| |
|---|
| If a sub-expression $t^{\ell}$ evaluates to a function (closure), then the function must be "predicted" by $\widehat{\mathsf{C}}(\ell)$ |

# CFA: Example

$$((\texttt{fn}\ x \texttt{ => } x^{1})^{2}\ (\texttt{fn}\ y \texttt{ => } y^{3})^{4})^{5}$$

| | $(\widehat{C}_e, \widehat{\rho}_e)$ | $(\widehat{C}'_e, \widehat{\rho}'_e)$ | $(\widehat{C}''_e, \widehat{\rho}''_e)$ |
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| 2 | $\{\texttt{fn}\ x \texttt{ => } x^{1}\}$ | $\{\texttt{fn}\ x \texttt{ => } x^{1}\}$ | $\{\texttt{fn}\ x \texttt{ => } x^{1}, \texttt{fn}\ y \texttt{ => } y^{3}\}$ |
| 3 | $\emptyset$ | $\emptyset$ | $\{\texttt{fn}\ x \texttt{ => } x^{1}, \texttt{fn}\ y \texttt{ => } y^{3}\}$ |
| 4 | $\{\texttt{fn}\ y \texttt{ => } y^{3}\}$ | $\{\texttt{fn}\ y \texttt{ => } y^{3}\}$ | $\{\texttt{fn}\ x \texttt{ => } x^{1}, \texttt{fn}\ y \texttt{ => } y^{3}\}$ |
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| 1 | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
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| 3 | $\emptyset$ | $\emptyset$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
| 4 | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
| 5 | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
| $x$ | $\{\texttt{fn}\ y \Rightarrow y^3\}$ | $\emptyset$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
| $y$ | $\emptyset$ | $\emptyset$ | $\{\texttt{fn}\ x \Rightarrow x^1, \texttt{fn}\ y \Rightarrow y^3\}$ |
| | $\surd$ | | $\surd$ |

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$$\begin{array}{l}(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (\texttt{let } x = t_1^{\ell_1} \texttt{ in } t_2^{\ell_2})^\ell \\ \quad \text{iff} \quad (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_1^{\ell_1} \wedge (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_2^{\ell_2} \wedge \\ \quad\quad\quad \widehat{\mathsf{C}}(\ell_1) \subseteq \widehat{\rho}(x) \wedge \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\mathsf{C}}(\ell)\end{array}$$

# Specification: Rules II

$$
\begin{aligned}
&(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (t_1^{\ell_1}\ op\ t_2^{\ell_2})^{\ell} \\
&\qquad \text{iff} \quad (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_2^{\ell_2}
\end{aligned}
$$

# Specification: Rules II

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (t_1^{\ell_1}\ op\ t_2^{\ell_2})^{\ell}$$
$$\text{iff} \quad (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_2^{\ell_2}$$

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (\texttt{fn}\ x\ \texttt{=>}\ e_0)^{\ell}$$
$$\text{iff} \quad \{\texttt{fn}\ x\ \texttt{=>}\ e_0\} \subseteq \widehat{\mathsf{C}}(\ell) \ \wedge (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s e_0$$

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$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (t_1^{\ell_1}\ t_2^{\ell_2})^\ell$$
$$\begin{aligned}\text{iff}\quad & (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_2^{\ell_2} \ \wedge \\ & (\forall (\texttt{fn}\ x \texttt{ => } t_0^{\ell_0}) \in \widehat{\mathsf{C}}(\ell_1) : \\ & \qquad \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\rho}(x) \ \wedge \\ & \qquad \widehat{\mathsf{C}}(\ell_0) \subseteq \widehat{\mathsf{C}}(\ell))\end{aligned}$$

# Constraint Generation

To implement the specification, we must generate a set of constraints from a given program. $\mathcal{C}_\star[\![e_\star]\!]$ is a set of constraints and conditional constraints of the form

$$lhs \subseteq rhs$$

$$\{t\} \subseteq rhs' \Rightarrow lhs \subseteq rhs$$

where *rhs* is of the form $\mathrm{C}(\ell)$ or $\mathrm{r}(x)$, and *lhs* is of the form $\mathrm{C}(\ell), \mathrm{r}(x)$, or $\{t\}$, and all occurrences of $t$ are of the form `fn` $x$ `=>` $e_0$.

# Constraint-Based CFA I

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (\texttt{fn}\ x \texttt{ => } e_0)^\ell$$
$$\text{iff} \quad \{\texttt{fn}\ x \texttt{ => } e_0\} \subseteq \widehat{\mathsf{C}}(\ell) \ \wedge (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s e_0$$

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$$\begin{aligned}(\widehat{\mathsf{C}}, \widehat{\rho}) \models_s (t_1^{\ell_1}\ t_2^{\ell_2})^\ell \quad \text{iff} \quad & (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_1^{\ell_1} \ \wedge \ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_s t_2^{\ell_2} \ \wedge \\ & (\forall (\texttt{fn}\ x \texttt{ => } t_0^{\ell_0}) \in \widehat{\mathsf{C}}(\ell_1) : \ \ \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\rho}(x) \ \wedge \\ & \qquad\qquad\qquad\qquad\qquad\qquad \widehat{\mathsf{C}}(\ell_0) \subseteq \widehat{\mathsf{C}}(\ell))\end{aligned}$$

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$$\widehat{\mathsf{C}}(\ell_0) \subseteq \widehat{\mathsf{C}}(\ell))$$

$$\begin{array}{|l|}\hline
\mathcal{C}_\star[\![(t_1^{\ell_1}\ t_2^{\ell_2})^\ell]\!] \\
\ = \mathcal{C}_\star[\![t_1^{\ell_1}]\!] \cup \mathcal{C}_\star[\![t_2^{\ell_2}]\!] \\
\ \cup \{\{t\} \subseteq \mathsf{C}(\ell_1) \Rightarrow \mathsf{C}(\ell_2) \subseteq \mathsf{r}(x) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \mathbf{Term}_\star\} \\
\ \cup \{\{t\} \subseteq \mathsf{C}(\ell_1) \Rightarrow \mathsf{C}(\ell_0) \subseteq \mathsf{C}(\ell) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \mathbf{Term}_\star\} \\
\hline\end{array}$$

# Constraint-Based CFA II

$$\mathcal{C}_\star[\![c^\ell]\!] = \emptyset$$

$$\mathcal{C}_\star[\![x^\ell]\!] = \{\mathsf{r}(x) \subseteq \mathsf{C}(\ell)\}$$

$$\begin{aligned}\mathcal{C}_\star[\![(\texttt{if } t_0^{\ell_0} \texttt{ then } t_1^{\ell_1} \texttt{ else } t_2^{\ell_2})^\ell]\!] \; &= \mathcal{C}_\star[\![t_0^{\ell_0}]\!] \cup \mathcal{C}_\star[\![t_1^{\ell_1}]\!] \cup \mathcal{C}_\star[\![t_2^{\ell_2}]\!] \\ &\cup \{\mathsf{C}(\ell_1) \subseteq \mathsf{C}(\ell)\} \\ &\cup \{\mathsf{C}(\ell_2) \subseteq \mathsf{C}(\ell)\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_\star[\![(\texttt{let } x = t_1^{\ell_1} \texttt{ in } t_2^{\ell_2})^\ell]\!] \; &= \mathcal{C}_\star[\![t_1^{\ell_1}]\!] \cup \mathcal{C}_\star[\![t_2^{\ell_2}]\!] \\ &\cup \{\mathsf{C}(\ell_1) \subseteq \mathsf{r}(x)\} \cup \{\mathsf{C}(\ell_2) \subseteq \mathsf{C}(\ell)\}\end{aligned}$$

$$\mathcal{C}_\star[\![(t_1^{\ell_1} \; op \; t_2^{\ell_2})^\ell]\!] \; = \mathcal{C}_\star[\![t_1^{\ell_1}]\!] \cup \mathcal{C}_\star[\![t_2^{\ell_2}]\!]$$

# Contraint Generation: Example I

$$\mathcal{C}_\star[\![((\texttt{fn}\ x\ \texttt{=>}\ x^{1})^{2}\ (\texttt{fn}\ y\ \texttt{=>}\ y^{3})^{4})^{5}]\!] =$$

$$\mathcal{C}_\star[\![(\texttt{fn}\ x\ \texttt{=>}\ x^{1})^{2}]\!] \cup \mathcal{C}_\star[\![(\texttt{fn}\ y\ \texttt{=>}\ y^{3})^{4}]\!]$$
$$\cup \{\{t\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(x) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \mathbf{Term}_\star\}$$
$$\cup \{\{t\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(\ell_0) \subseteq \mathsf{C}(5) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \mathbf{Term}_\star\}$$

$$\mathcal{C}_\star[\![(\texttt{fn}\ x\ \texttt{=>}\ x^{1})^{2}]\!] =$$
$$\{\{\texttt{fn}\ x\ \texttt{=>}\ x^{1}\} \subseteq \mathsf{C}(2)\} \cup \mathcal{C}_\star[\![x^{1}]\!] =$$
$$\{\{\texttt{fn}\ x\ \texttt{=>}\ x^{1}\} \subseteq \mathsf{C}(2)\} \cup \{\mathsf{r}(\texttt{x}) \subseteq \mathsf{C}(1)\} =$$
$$\{\{\texttt{fn}\ x\ \texttt{=>}\ x^{1}\} \subseteq \mathsf{C}(2), \mathsf{r}(\texttt{x}) \subseteq \mathsf{C}(1)\}$$

$$\mathcal{C}_\star[\![(\texttt{fn}\ y\ \texttt{=>}\ y^{3})^{4}]\!] = \{\{\texttt{fn}\ y\ \texttt{=>}\ y^{3}\} \subseteq \mathsf{C}(4)\} \cup \mathcal{C}_\star[\![y^{3}]\!] =$$
$$\{\{\texttt{fn}\ y\ \texttt{=>}\ y^{3}\} \subseteq \mathsf{C}(4), \mathsf{r}(\texttt{y}) \subseteq \mathsf{C}(3)\}$$

# Contraint Generation: Example II

$$\{\{t\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(x) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \textbf{Term}_\star\}$$

$$= \{\ \texttt{fn}\ x\ \texttt{=>}\ x^1 \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(\texttt{x}),$$
$$\texttt{fn}\ y\ \texttt{=>}\ y^3 \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(\texttt{y})\ \}$$

$$\{\{t\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(\ell_0) \subseteq \mathsf{C}(5) \mid t = (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \textbf{Term}_\star\}$$

$$= \{\ \texttt{fn}\ x\ \texttt{=>}\ x^1 \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(1) \subseteq \mathsf{C}(5),$$
$$\texttt{fn}\ y\ \texttt{=>}\ y^3 \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(3) \subseteq \mathsf{C}(5)\ \}$$

# Contraint Generation: Example III

$$\mathcal{C}_\star[\![((\texttt{fn}\ x \texttt{ => } x^{1})^{2}\ (\texttt{fn}\ y \texttt{ => } y^{3})^{4})^{5}]\!] =$$

$$\begin{aligned}
\{ & \{\ \texttt{fn}\ x \texttt{ => } x^{1}\} \subseteq \mathsf{C}(2), \\
& \mathsf{r}(\texttt{x}) \subseteq \mathsf{C}(1), \\
& \{\texttt{fn}\ y \texttt{ => } y^{3}\} \subseteq \mathsf{C}(4), \\
& \mathsf{r}(\texttt{y}) \subseteq \mathsf{C}(3), \\
& \{\texttt{fn}\ x \texttt{ => } x^{1}\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(\texttt{x}), \\
& \{\texttt{fn}\ x \texttt{ => } x^{1}\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(1) \subseteq \mathsf{C}(5), \\
& \{\texttt{fn}\ y \texttt{ => } y^{3}\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(4) \subseteq \mathsf{r}(\texttt{y}), \\
& \{\texttt{fn}\ y \texttt{ => } y^{3}\} \subseteq \mathsf{C}(2) \Rightarrow \mathsf{C}(3) \subseteq \mathsf{C}(5)\ \}
\end{aligned}$$

# Constraint Solving

To solve the constraints, we use a graph-based formulation.
The algorithm uses the following main data structures:

- a worklist W, i.e. a list of nodes whose outgoing edges should be traversed;
- a data array D that for each node gives an element of $\widehat{\mathbf{Val}}_\star$; and
- an edge array E that for each node gives a list of constraints from which a list of the successor nodes can be computed.

# Constraints Graph

The graph will have nodes $\mathsf{C}(\ell)$ and $\mathsf{r}(x)$ for $\ell \in \mathbf{Lab}_\star$ and $x \in \mathbf{Var}_\star$. Associated with each node $p$ we have a data field D[p] that initially is given by:

$$\mathrm{D[p]} = \{t \mid (\{t\} \subseteq p) \in \mathcal{C}_\star[\![e_\star]\!]\}$$

The graph will have edges for a subset of the constraints in $\mathcal{C}_\star[\![e_\star]\!]$; each edge will be decorated with the constraint that gives rise to it:

- a constraint $p_1 \subseteq p_2$ gives rise to an edge from $p_1$ to $p_2$, and
- a constraint $\{t\} \subseteq p \Rightarrow p_1 \subseteq p_2$ gives rise to an edge from $p_1$ to $p_2$ *and* an edge from $p$ to $p_2$.

# Algorithm I

INPUT: $\mathcal{C}_\star[\![e_\star]\!]$

OUTPUT: $(\widehat{\mathsf{C}}, \widehat{\rho})$

METHOD: Step 1: Initialisation

W := nil;
for $q$ in Nodes do D[$q$] := $\emptyset$;
for $q$ in Nodes do E[$q$] := nil;

# Algorithm II

Step 2: Building the graph

for $cc$ in $\mathcal{C}_\star[\![e_\star]\!]$ do
  case $cc$ of
    $\{t\} \subseteq p$: add($p$,$\{t\}$);
    $p_1 \subseteq p_2$: E[$p_1$] := cons($cc$,E[$p_1$]);
    $\{t\} \subseteq p \Rightarrow p_1 \subseteq p_2$:
      E[$p_1$] := cons($cc$,E[$p_1$]);
      E[$p$] := cons($cc$,E[$p$]);

# Algorithm III

Step 3: Iteration

```
while W ≠ nil do
    q := head(W); W := tail(W);
    for cc in E[q] do
        case cc of
            p1 ⊆ p2: add(p2, D[p1]);
            {t} ⊆ p ⇒ p1 ⊆ p2:
                if t ∈ D[p] then add(p2, D[p1]);
```

# Algorithm IV

Step 4: Recording the solution

for $\ell$ in $\mathbf{Lab}_\star$ do $\widehat{\mathsf{C}}(\ell)$ := $\mathsf{D}[\mathsf{C}(\ell)]$;
for $x$ in $\mathbf{Var}_\star$ do $\widehat{\rho}(x)$ := $\mathsf{D}[\mathsf{r}(x)]$;

USING: procedure add($q$,$d$) is
if $\neg$ ($d \subseteq$ D[$q$])
then D[$q$] := D[$q$] $\cup$ $d$;
W := cons($q$,W);

# Example I

| $p$ | D[$p$] | E[$p$] |
|---|---|---|
| C(1) | $\emptyset$ | [$\text{id}_x \subseteq$C(2) $\Rightarrow$ C(1)$\subseteq$C(5)] |
| C(2) | $\text{id}_x$ | [$\text{id}_y \subseteq$C(2) $\Rightarrow$ C(3)$\subseteq$C(5), $\text{id}_y \subseteq$C(2) $\Rightarrow$ C(4)$\subseteq$r($\mathtt{y}$), $\text{id}_x \subseteq$C(2) $\Rightarrow$ C(1)$\subseteq$C(5), $\text{id}_x \subseteq$C(2) $\Rightarrow$ C(4)$\subseteq$r($\mathtt{x}$)] |
| C(3) | $\emptyset$ | [$\text{id}_y \subseteq$C(2) $\Rightarrow$ C(3)$\subseteq$C(5)] |
| C(4) | $\text{id}_y$ | [$\text{id}_y \subseteq$C(2) $\Rightarrow$ C(4)$\subseteq$r($\mathtt{y}$), $\text{id}_x \subseteq$C(2) $\Rightarrow$ C(4)$\subseteq$r($\mathtt{x}$)] |
| C(5) | $\emptyset$ | [ ] |
| r($\mathtt{x}$) | $\emptyset$ | [r($\mathtt{x}$)$\subseteq$C(1)] |
| r($\mathtt{y}$) | $\emptyset$ | [r($\mathtt{y}$)$\subseteq$C(3)] |

# Example II

| W | [C(4),C(2)] | [r($\mathtt{x}$),C(2)] | [C(1),C(2)] | [C(5),C(2)] | [C(2)] | [ ] |
|---|---|---|---|---|---|---|
| C(1) | $\emptyset$ | $\emptyset$ | $id_y$ | $id_y$ | $id_y$ | $id_y$ |
| C(2) | $id_x$ | $id_x$ | $id_x$ | $id_x$ | $id_x$ | $id_x$ |
| C(3) | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| C(4) | $id_y$ | $id_y$ | $id_y$ | $id_y$ | $id_y$ | $id_y$ |
| C(5) | $\emptyset$ | $\emptyset$ | $\emptyset$ | $id_y$ | $id_y$ | $id_y$ |
| r($\mathtt{x}$) | $\emptyset$ | $id_y$ | $id_y$ | $id_y$ | $id_y$ | $id_y$ |
| r($\mathtt{y}$) | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

# Control Flow + Data Flow

Let **Data** be a set of *abstract data values* (i.e. abstract properties of booleans and arithmetic constants)

$$\widehat{v} \in \widehat{\mathbf{Val}}_d = \mathcal{P}(\mathbf{Term} \cup \mathbf{Data}) \quad \text{abstract values}$$

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For each constant $c \in \mathbf{Const}$ we need an element $d_c \in \mathbf{Data}$
Similarly, for each operator $op \in \mathbf{Op}$ we need a total function

$$\widehat{\mathrm{op}} : \widehat{\mathbf{Val}}_d \times \widehat{\mathbf{Val}}_d \to \widehat{\mathbf{Val}}_d$$

# Control Flow + Data Flow

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For each constant $c \in \mathbf{Const}$ we need an element $d_c \in \mathbf{Data}$
Similarly, for each operator $op \in \mathbf{Op}$ we need a total function

$$\widehat{\mathrm{op}} : \widehat{\mathbf{Val}}_d \times \widehat{\mathbf{Val}}_d \to \widehat{\mathbf{Val}}_d$$

Typically, $\widehat{\mathrm{op}}$ will have a definition of the form:

$$\widehat{v}_1 \ \widehat{\mathrm{op}} \ \widehat{v}_2 = \bigcup \{ d_{op}(d_1, d_2) \mid d_1 \in \widehat{v}_1 \cap \mathbf{Data}, d_2 \in \widehat{v}_2 \cap \mathbf{Data} \}$$

for some function $d_{op} : \mathbf{Data} \times \mathbf{Data} \to \mathcal{P}(\mathbf{Data})$

# Detection of Sign

$$\mathbf{Data}_{\mathsf{sign}} = \{\mathsf{tt}, \mathsf{ff}, -, 0, +\}$$

$$d_{\texttt{true}} = \mathsf{tt} \qquad d_7 = +$$

# Detection of Sign

$$\mathbf{Data}_{\text{sign}} = \{\text{tt}, \text{ff}, -, 0, +\}$$

$$d_{\texttt{true}} = \text{tt} \qquad d_7 = +$$

$\widehat{+}$ is defined from:

| $d_+$ | tt | ff | $-$ | $0$ | $+$ |
|---|---|---|---|---|---|
| tt | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| ff | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $-$ | $\emptyset$ | $\emptyset$ | $\{-\}$ | $\{-\}$ | $\{-, 0, +\}$ |
| $0$ | $\emptyset$ | $\emptyset$ | $\{-\}$ | $\{0\}$ | $\{+\}$ |
| $+$ | $\emptyset$ | $\emptyset$ | $\{-, 0, +\}$ | $\{+\}$ | $\{+\}$ |

# Abstract Values I

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (\texttt{fn}\ x\ \texttt{=>}\ e_0)^\ell \text{ iff } \{\texttt{fn}\ x\ \texttt{=>}\ e_0\} \subseteq \widehat{\mathsf{C}}(\ell) \wedge (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d e_0$$

$$\begin{array}{ll} (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (t_1^{\ell_1}\ t_2^{\ell_2})^\ell & \\ \quad \text{iff} & (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_2^{\ell_2} \wedge \\ & (\forall (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \widehat{\mathsf{C}}(\ell_1) : \\ & \qquad \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\rho}(x) \ \wedge\ \widehat{\mathsf{C}}(\ell_0) \subseteq \widehat{\mathsf{C}}(\ell)) \end{array}$$

# Abstract Values I

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (\texttt{fn}\ x\ \texttt{=>}\ e_0)^{\ell} \text{ iff } \{\texttt{fn}\ x\ \texttt{=>}\ e_0\} \subseteq \widehat{\mathsf{C}}(\ell) \wedge (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d e_0$$

$$\begin{array}{ll}(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (t_1^{\ell_1}\ t_2^{\ell_2})^{\ell} & \\ \quad \text{iff} & (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_2^{\ell_2} \wedge \\ & (\forall (\texttt{fn}\ x\ \texttt{=>}\ t_0^{\ell_0}) \in \widehat{\mathsf{C}}(\ell_1) : \\ & \qquad \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\rho}(x)\ \wedge\ \widehat{\mathsf{C}}(\ell_0) \subseteq \widehat{\mathsf{C}}(\ell))\end{array}$$

$$\begin{array}{ll}(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (\texttt{if}\ t_0^{\ell_0}\ \texttt{then}\ t_1^{\ell_1}\ \texttt{else}\ t_2^{\ell_2})^{\ell} & \\ \quad \text{iff} & (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_0^{\ell_0} \wedge \\ & (d_{\texttt{true}} \in \widehat{\mathsf{C}}(\ell_0) \Rightarrow ((\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_1^{\ell_1}\ \wedge\ \widehat{\mathsf{C}}(\ell_1) \subseteq \widehat{\mathsf{C}}(\ell))) \wedge \\ & (d_{\texttt{false}} \in \widehat{\mathsf{C}}(\ell_0) \Rightarrow ((\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_2^{\ell_2}\ \wedge\ \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\mathsf{C}}(\ell)))\end{array}$$

# Abstract Values II

$$(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d c^\ell \text{ iff } \{d_c\} \subseteq \widehat{\mathsf{C}}(\ell)$$

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$$\begin{array}{l} (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (\texttt{let } x = t_1^{\ell_1} \texttt{ in } t_2^{\ell_2})^\ell \\ \qquad \text{iff} \quad (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_1^{\ell_1} \ \wedge \ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_2^{\ell_2} \ \wedge \\ \qquad \qquad \widehat{\mathsf{C}}(\ell_1) \subseteq \widehat{\rho}(x) \ \wedge \ \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\mathsf{C}}(\ell) \end{array}$$

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$$\begin{array}{l}(\widehat{\mathsf{C}}, \widehat{\rho}) \models_d (t_1^{\ell_1}\ op\ t_2^{\ell_2})^\ell \\ \qquad \text{iff} \quad (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_1^{\ell_1} \ \wedge\ (\widehat{\mathsf{C}}, \widehat{\rho}) \models_d t_2^{\ell_2} \ \wedge \\ \qquad\qquad\ \ \widehat{\mathsf{C}}(\ell_1)\ \widehat{\mathsf{op}}\ \widehat{\mathsf{C}}(\ell_2) \subseteq \widehat{\mathsf{C}}(\ell)\end{array}$$

# Example: Sign Detection

$$\begin{array}{lll}\texttt{let} & f = (\texttt{fn}\ x \texttt{=>} & (\texttt{if}\ (x^{1} > 0^{2})^{3}\ \texttt{then}\ (\texttt{fn}\ y \texttt{=>}\ y^{4})^{5} \\ & & \texttt{else}\ (\texttt{fn}\ z \texttt{=>}\ 25^{6})^{7})^{8})^{9} \\ \texttt{in} & ((f^{10}\,3^{11})^{12}\,0^{13})^{14})^{15} & \end{array}$$

# Example: Sign Detection

$$
\begin{array}{lll}
\texttt{let} & f = (\texttt{fn}\ x \texttt{=>} & (\texttt{if}\ (x^{1} > 0^{2})^{3}\ \texttt{then}\ (\texttt{fn}\ y \texttt{=>}\ y^{4})^{5} \\
 & & \texttt{else}\ (\texttt{fn}\ z \texttt{=>}\ 25^{6})^{7})^{8})^{9} \\
\texttt{in} & ((f^{10}3^{11})^{12}0^{13})^{14})^{15} &
\end{array}
$$

| | |
|---|---|
| C(1) | $\emptyset$ |
| C(2) | $\emptyset$ |
| C(3) | $\emptyset$ |
| C(4) | $\emptyset$ |
| C(5) | $id_y$ |
| C(6) | $\emptyset$ |
| C(7) | $c_{25}$ |

| | |
|---|---|
| C(8) | $\{id_y, c_{25}\}$ |
| C(9) | {fn x ...)$^{8}$} |
| C(10) | {fn x ...)$^{8}$} |
| C(11) | $\emptyset$ |
| C(12) | $\{id_y, c_{25}\}$ |
| C(13) | $\emptyset$ |

| | |
|---|---|
| C(14) | $\emptyset$ |
| C(15) | $\emptyset$ |
| r(f) | {fn x ...)$^{8}$} |
| r(x) | $\emptyset$ |
| r(y) | $\emptyset$ |
| r(z) | $\emptyset$ |

# Example: Sign Detection

$$\begin{array}{ll} \texttt{let} & f = (\texttt{fn}\ x \texttt{=>} \quad (\texttt{if}\ (x^{1} > 0^{2})^{3}\ \texttt{then}\ (\texttt{fn}\ y \texttt{=>}\ y^{4})^{5} \\ & \qquad\qquad\qquad \texttt{else}\ (\texttt{fn}\ z \texttt{=>}\ 25^{6})^{7})^{8})^{9} \\ \texttt{in} & ((f^{10} 3^{11})^{12} 0^{13})^{14})^{15} \end{array}$$

| | |
|---|---|
| C(1) | $\{+\}$ |
| C(2) | $\{0\}$ |
| C(3) | $\{\texttt{tt}\}$ |
| C(4) | $\{0\}$ |
| C(5) | $id_y$ |
| C(6) | $\emptyset$ |
| C(7) | $c_{25}$ |

| | |
|---|---|
| C(8) | $\{id_y\}$ |
| C(9) | $\{\texttt{fn x ...)}^{8}\}$ |
| C(10) | $\{\texttt{fn x ...)}^{8}\}$ |
| C(11) | $\{+\}$ |
| C(12) | $\{id_y\}$ |
| C(13) | $\{0\}$ |

| | |
|---|---|
| C(14) | $\{0\}$ |
| C(15) | $\{0\}$ |
| r(f) | $\{\texttt{fn x ...)}^{8}\}$ |
| r(x) | $\{+\}$ |
| r(y) | $\{0\}$ |
| r(z) | $\emptyset$ |

# Example: Sign Detection

$$
\begin{array}{ll}
\texttt{let} & f = (\texttt{fn}\ x \texttt{ => } (\texttt{if}\ (x^{1} > 0^{2})^{3}\ \texttt{then}\ (\texttt{fn}\ y \texttt{ => } y^{4})^{5} \\
 & \qquad\qquad\quad \texttt{else}\ (\texttt{fn}\ z \texttt{ => } 25^{6})^{7})^{8})^{9} \\
\texttt{in} & ((f^{10}3^{11})^{12}0^{13})^{14})^{15}
\end{array}
$$

A pure 0-CFA analysis will not be able to discover that the `else`-branch of the conditional will never be executed.

# Example: Sign Detection

$$\begin{array}{lll} \texttt{let} & f = (\texttt{fn}\ x \Rightarrow & (\texttt{if}\ (x^{1} > 0^{2})^{3}\ \texttt{then}\ (\texttt{fn}\ y \Rightarrow y^{4})^{5} \\ & & \texttt{else}\ (\texttt{fn}\ z \Rightarrow 25^{6})^{7})^{8})^{9} \\ \texttt{in} & ((f^{10}\,3^{11})^{12}\,0^{13})^{14})^{15} & \end{array}$$

A pure 0-CFA analysis will not be able to discover that the `else`-branch of the conditional will never be executed.

When we combine the analysis with a Detection of Signs Analysis then the analysis can determine that only $\texttt{fn}\ y \Rightarrow y^{4}$ is a possible abstraction at label 12.

# Context-Sensitive CFA

The Control Flow Analyses presented so far are imprecise in that they cannot distinguish the various instances of function calls from one another. In the terminology of Data Flow Analysis the 0-CFA analysis is context-insensitive and in the terminology of Control Flow Analysis it is monovariant.

# Context-Sensitive CFA

The Control Flow Analyses presented so far are imprecise in that they cannot distinguish the various instances of function calls from one another. In the terminology of Data Flow Analysis the 0-CFA analysis is context-insensitive and in the terminology of Control Flow Analysis it is monovariant.

To get a more precise analysis it is useful to introduce a mechanism that distinguishes different dynamic instances of variables and labels from one another. This results in a context-sensitive analysis and in the terminology of Control Flow Analysis the term polyvariant is used.

# Example: Context

Consider the expression:

$$\begin{array}{ll}(\texttt{let} & f = (\texttt{fn}\; x \texttt{=>}\; x^{1})^{2} \\ \texttt{in} & ((f^{3}\; f^{4})^{5}\; (\texttt{fn}\; y \texttt{=>}\; y^{6})^{7})^{8})^{9}\end{array}$$

The least 0-CFA analysis is given by $(\widehat{\mathsf{C}}_{\mathsf{id}}, \widehat{\rho}_{\mathsf{id}})$:

# 0-CFA Solutions

$\widehat{\mathsf{C}}_{\mathsf{id}}(1) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}, \texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(2) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(3) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(4) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(5) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}, \texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(6) = \{\texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(7) = \{\texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(8) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}, \texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\mathsf{C}}_{\mathsf{id}}(9) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}, \texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\rho}_{\mathsf{id}}(\texttt{f}) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}\}$

$\widehat{\rho}_{\mathsf{id}}(\texttt{x}) = \{\texttt{fn}\ x\texttt{=>}\ x^{1}, \texttt{fn}\ y \texttt{=>}\ y^{6}\}$

$\widehat{\rho}_{\mathsf{id}}(\texttt{y}) = \{\texttt{fn}\ y \texttt{=>}\ y^{6}\}$

# Expansion

Expand the program into

$$\begin{array}{ll} \texttt{let} & f_1 = (\texttt{fn}\ x_1 \texttt{ => } x_1) \\ \texttt{in} & \texttt{let}\ f_2 = (\texttt{fn}\ x_2 \texttt{ => } x_2) \\ & \quad \texttt{in}\ (f_1\ f_2)\ (\texttt{fn}\ y \texttt{ => } y) \end{array}$$

and then analyse the expanded expression: the 0-CFA analysis is now able to deduce that $x_1$ can only be bound to $\texttt{fn}\ x_2 \texttt{ => } x_2$ and that $x_2$ can only be bound to $\texttt{fn}\ y \texttt{ => } y$ so the overall expression will evaluate to $\texttt{fn}\ y \texttt{ => } y$ only.

# Further CFA Analyses

A more satisfactory solution to the problem is to extend the analysis with context information allowing it to distinguish between the various instances of variables and program points and still analyse the original expression.

Examples of such analyses include $k$-CFA analyses, uniform $k$-CFA analyses, polynomial $k$-CFA analyses (mainly of interest for $k > 0$) and the Cartesian Product Algorithm.