Program Analysis (70020)

Data Flow Analysis

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Data Flow Analysis

The general approach for determining program properties for procedural languages via a dataflow analysis:

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- Extract (Control) Flow Information

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- Formulate Data Flow Equations

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- Construct Solution(s) of Equations

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The general approach for determining program properties for procedural languages via a dataflow analysis:

- Extract (Control) Flow Information
- Formulate Data Flow Equations
  - Update Local Information
  - Collect Global Information
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Available Expressions

The *Available Expressions Analysis* will determine:

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This information can be used to avoid the re-computation of an expression. For clarity, we will concentrate on arithmetic expressions.

Example

Consider the following simple program:

$$
\begin{array}{l}
[\ x := a + b\ ]^1; \\
[\ y := a * b\ ]^2; \\
\textbf{while}\ [y > a + b]^3\ \textbf{do}\ (\\
\qquad [\ a := a + 1\ ]^4; \\
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\end{array}
$$

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$$

It should be clear that the expression a+b is available every time the execution reaches the test (label 3) in the loop; as a consequence, the expression need not be recomputed.

AE Analysis

$$\textit{kill}_{\mathsf{AE}} : \textbf{Block}_\star \rightarrow \mathcal{P}(\textbf{AExp}_\star)$$

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AE Auxiliary Functions

$$\begin{array}{rcl} kill_{\mathsf{AE}}([\, x := a \,]^{\ell}) & = & \{a' \in \mathbf{AExp}_{\star} \mid x \in FV(a')\} \\ kill_{\mathsf{AE}}([\, \mathbf{skip} \,]^{\ell}) & = & \emptyset \\ kill_{\mathsf{AE}}([b]^{\ell}) & = & \emptyset \end{array}$$

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gen_{\mathsf{AE}}([\, x := a \,]^{\ell}) & = & \{a' \in \mathbf{AExp}(a) \mid x \notin FV(a')\} \\
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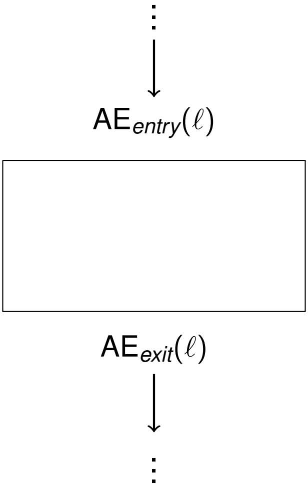
gen_{\mathsf{AE}}([b]^{\ell}) & = & \mathbf{AExp}(b)
\end{array}
$$

AE Local Change (e.g. expression $x + y$)



AE Local Change (e.g. expression $x + y$)

$\vdots$

$\downarrow$

$\text{AE}_{entry}(\ell)$

$[\ z := x + y\ ]^{\ell}\ \surd$

$\text{AE}_{exit}(\ell)$

$\downarrow$

$\vdots$

AE Local Change (e.g. expression $x + y$)

$$\vdots$$

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$$[x + y < x]^{\ell} \quad \surd$$

$$\mathrm{AE}_{exit}(\ell)$$

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$$\vdots$$

AE Local Change (e.g. expression $x + y$)

$$\vdots$$

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$$[\; {\color{red} x} := x + y \;]^{\ell} \quad {\color{red}\times}$$

$$\mathrm{AE}_{exit}(\ell)$$

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$$\vdots$$

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$$[x + y < x]^{\ell} \quad \surd$$
$$[\ z := x + y\ ]^{\ell} \quad \surd$$
$$[\ x := x + y\ ]^{\ell} \quad \times$$

$$\text{AE}_{exit}(\ell)$$

$$\downarrow$$

$$\vdots$$

Whenever a variable x in an expression gets a new value the expression becomes unavailable.

AE Equation Schemes

$$\mathsf{AE}_{\mathit{entry}}(\ell) \;=\; \begin{cases} \emptyset, \text{ if } \ell = \mathit{init}(S_\star) \\ \bigcap\{\mathsf{AE}_{\mathit{exit}}(\ell') \mid (\ell', \ell) \in \mathit{flow}(S_\star)\}, \text{ otherwise} \end{cases}$$

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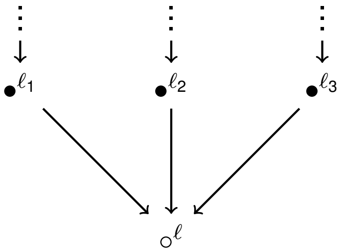
$$\mathsf{AE}_{exit}(\ell) \;=\; (\mathsf{AE}_{entry}(\ell) \backslash kill_{\mathsf{AE}}([B]^\ell)) \cup gen_{\mathsf{AE}}(([B]^\ell) \text{ where } [B]^\ell \in blocks(S_\star)$$

AE Global Collection

$\circ^{\ell}$

AE Global Collection



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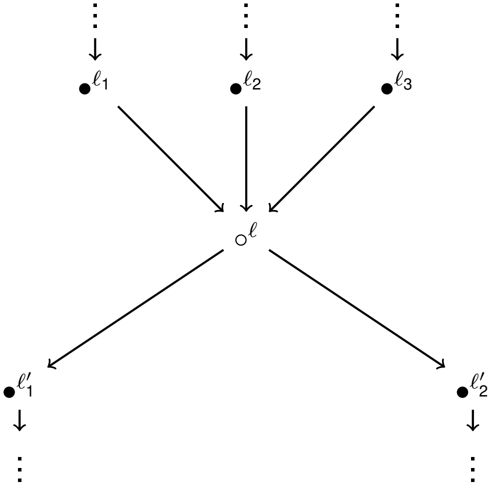

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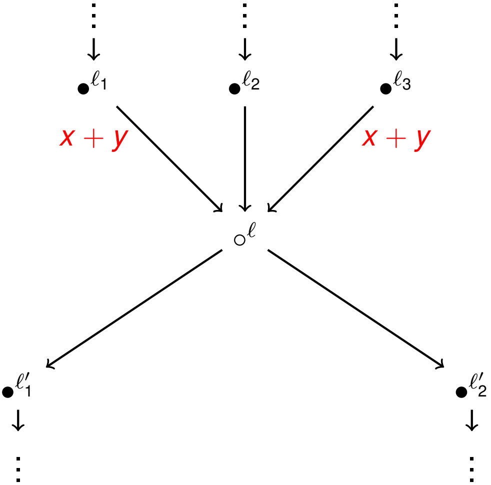

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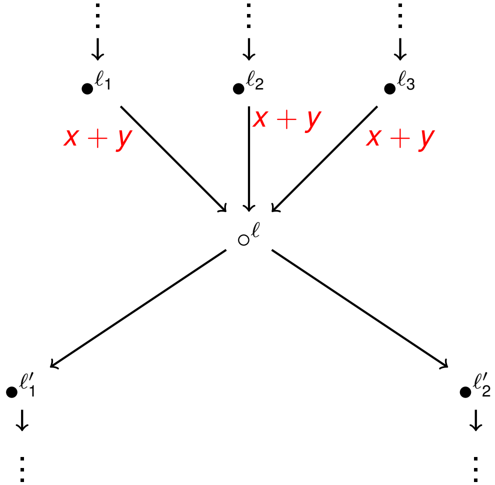

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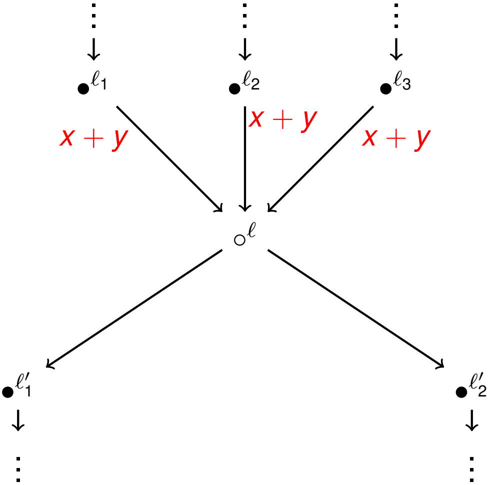

We push information “forward in time”.

Largest Solution

The analysis is a *forward analysis* and we are interested in the *largest* sets satisfying the equation for AE_{entry} and AE_{exit}.

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The analysis is a *forward analysis* and we are interested in the *largest* sets satisfying the equation for AE_{entry} and AE_{exit}.

$$[\ z := x + y\ ]^{\ell};\textbf{while}\ [true]^{\ell'}\ \textbf{do}\ [\ \textbf{skip}\ ]^{\ell''}$$

$$\begin{array}{rcl}
\mathsf{AE}_{entry}(\ell) & = & \emptyset \\
\mathsf{AE}_{entry}(\ell') & = & \mathsf{AE}_{exit}(\ell)\ \cap\ \mathsf{AE}_{exit}(\ell'') \\
\mathsf{AE}_{entry}(\ell'') & = & \mathsf{AE}_{exit}(\ell') \\
\mathsf{AE}_{exit}(\ell) & = & \mathsf{AE}_{entry}(\ell) \cup \{x + y\} \\
\mathsf{AE}_{exit}(\ell') & = & \mathsf{AE}_{entry}(\ell') \\
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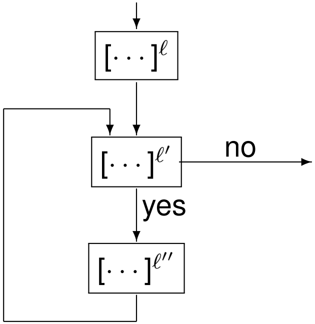
\end{array}$$

Obtaining Solutions



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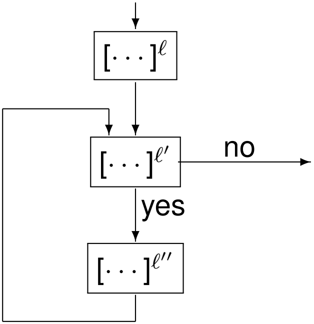

After some simplification, we find that:

$$\mathsf{AE}_{entry}(\ell') = \{x + y\} \cap \mathsf{AE}_{entry}(\ell')$$

AE Example

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ℓ	$kill_{\mathsf{AE}}(\ell)$	$gen_{\mathsf{AE}}(\ell)$
1	$\emptyset$	$\{a + b\}$
2	$\emptyset$	$\{a * b\}$
3	$\emptyset$	$\{a + b\}$
4	$\{a + b, a * b, a + 1\}$	$\emptyset$
5	$\emptyset$	$\{a + b\}$

AE Example: Equations

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\begin{array}{l}
[\ x := a + b\ ]^1; \\
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$$
\begin{aligned}
\mathrm{AE}_{exit}(1) &= \mathrm{AE}_{entry}(1) \cup \{a + b\}\\
\mathrm{AE}_{exit}(2) &= \mathrm{AE}_{entry}(2) \cup \{a * b\}\\
\mathrm{AE}_{exit}(3) &= \mathrm{AE}_{entry}(3) \cup \{a + b\}\\
\mathrm{AE}_{exit}(4) &= \mathrm{AE}_{entry}(4) \backslash \{a + b, a * b, a + 1\}\\
\mathrm{AE}_{exit}(5) &= \mathrm{AE}_{entry}(5) \cup \{a + b\}
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\begin{array}{rcl}
\mathrm{AE}_{entry}(1) & = & \emptyset \\
\mathrm{AE}_{entry}(2) & = & \mathrm{AE}_{exit}(1) \\
\mathrm{AE}_{entry}(3) & = & \mathrm{AE}_{exit}(2) \cap \mathrm{AE}_{exit}(5) \\
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\mathrm{AE}_{exit}(1) &= \mathrm{AE}_{entry}(1) \cup \{a+b\} \\
\mathrm{AE}_{exit}(2) &= \mathrm{AE}_{entry}(2) \cup \{a*b\} \\
\mathrm{AE}_{exit}(3) &= \mathrm{AE}_{entry}(3) \cup \{a+b\} \\
\mathrm{AE}_{exit}(4) &= \mathrm{AE}_{entry}(4) \backslash \{a+b, a*b, a+1\} \\
\mathrm{AE}_{exit}(5) &= \mathrm{AE}_{entry}(5) \cup \{a+b\}
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AE Example: Solutions

ℓ	$\mathsf{AE}_{entry}(\ell)$	$\mathsf{AE}_{exit}(\ell)$
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2	$\{a + b\}$	$\{a + b, a * b\}$
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$[\ x := a+b\ ]^1;$
$[\ y := a*b\ ]^2;$
while $[y > a+b]^3$ **do** (
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AE Example: Solutions

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Note that, even though a is redefined in the loop, the expression a+b is re-evaluated in the loop and so it is always available on entry to the loop. On the other hand, a*b is available on the first entry to the loop but is killed before the next iteration.

Reaching Definitions Analysis

The *Reaching Definitions Analysis* is analogous to the previous one except that we are interested in:

> *For each program point, which assignments* may *have been made and not overwritten, when program execution reaches this point along some path.*

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A main application of Reaching Definitions Analysis is in the construction of direct links between blocks that produce values and blocks that use them.

Example

A simple example to illustrate the *RD* analysis would be:

$$
\begin{array}{l}
[\ x := 5\ ]^1; \\
[\ y := 1\ ]^2; \\
\textbf{while}\ [x > 1]^3\ \textbf{do}\ (\\
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All of the assignments reach the entry of 4 (the assignments labelled 1 and 2 reach there on the first iteration); only the assignments labelled 1, 4 and 5 reach the entry of 5.

RD Analysis

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Remark: Strictly speaking we need $\mathcal{P}(\mathbf{Var}_\star \times (\mathbf{Lab}_\star \cup \{?\}))$.

RD Auxiliary Functions

$$
\begin{array}{rcl}
\mathit{kill}_{\mathrm{RD}}([\ x := a\ ]^{\ell}) & = & \{(x,?)\} \cup \{(x,\ell') \mid \\
 & & [B]^{\ell'} \text{ a "definition" of } x \text{ in } S_\star\} \\
\mathit{kill}_{\mathrm{RD}}([\ \textbf{skip}\ ]^{\ell}) & = & \emptyset \\
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\mathit{gen}_{\mathrm{RD}}([\, x := a \,]^{\ell}) &= \{(x, \ell)\} \\
\mathit{gen}_{\mathrm{RD}}([\, \textbf{skip} \,]^{\ell}) &= \emptyset \\
\mathit{gen}_{\mathrm{RD}}([b]^{\ell}) &= \emptyset
\end{aligned}
$$

RD Equation Schemes

$$\mathrm{RD}_{entry}(\ell) = \begin{cases} \{(x, ?) \mid x \in FV(S_\star)\}, & \text{if } \ell = init(S_\star) \\ \bigcup\{\mathrm{RD}_{exit}(\ell') \mid (\ell', \ell) \in flow(S_\star)\}, & \text{otherwise} \end{cases}$$

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$$\begin{aligned} \mathsf{RD}_{exit}(\ell) &= (\mathsf{RD}_{entry}(\ell) \backslash kill_{\mathsf{RD}}([B]^\ell)) \cup gen_{\mathsf{RD}}([B]^\ell) \\ &\quad \text{where } [B]^\ell \in blocks(S_\star) \end{aligned}$$

Smallest Solution

Similar to before, this is a *forward analysis* but we are interested in the *smallest* sets satisfying the equation for RD_{entry}.

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$$[\ z := x + y\ ]^{\ell}; \textbf{while}\ [true]^{\ell'}\ \textbf{do}\ [\ \textbf{skip}\ ]^{\ell''}$$

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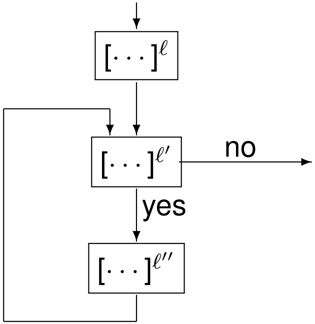
$$[\ z := x + y\ ]^{\ell};\textbf{while}\ [true]^{\ell'}\ \textbf{do}\ [\ \textbf{skip}\ ]^{\ell''}$$

$$
\begin{aligned}
\mathsf{RD}_{entry}(\ell) &= \{(x,?),(y,?),(z,?)\}\\
\mathsf{RD}_{entry}(\ell') &= \mathsf{RD}_{exit}(\ell) \cup \mathsf{RD}_{exit}(\ell'')\\
\mathsf{RD}_{entry}(\ell'') &= \mathsf{RD}_{exit}(\ell')\\
\mathsf{RD}_{exit}(\ell) &= (\mathsf{RD}_{entry}(\ell)\setminus\{(z,?)\}) \cup \{(z,\ell)\}\\
\mathsf{RD}_{exit}(\ell') &= \mathsf{RD}_{entry}(\ell')\\
\mathsf{RD}_{exit}(\ell'') &= \mathsf{RD}_{entry}(\ell'')
\end{aligned}
$$

Obtaining Solutions



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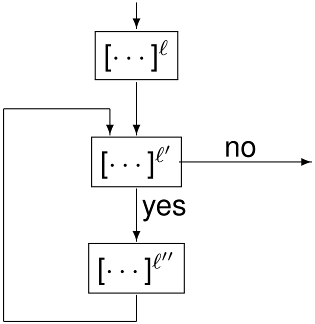

After some simplification, we find that:

$$\mathsf{RD}_{entry}(\ell') = \{(x, ?), (y, ?), (z, \ell)\} \cup \mathsf{RD}_{entry}(\ell')$$

RD Variations

Sometimes, when the Reaching Definitions analysis is presented in the literature, one has $\mathrm{RD}_{entry}(init(S_\star)) = \emptyset$ rather than $\mathrm{RD}_{entry}(init(S_\star)) = \{((x, ?) \mid x \in FV(S_\star)\}$.

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This is correct only for programs that always assign variables before their first use; incorrect optimisations may result if this is not the case. The advantage of our formulation is that it is always semantically sound.

RD Example

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[\ x := 5\ ]^1; \\
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ℓ	$kill_{\mathsf{RD}}(\ell)$	$gen_{\mathsf{RD}}(\ell)$
1	$\{(x,?),(x,1),(x,5)\}$	$\{(x,1)\}$
2	$\{(y,?),(y,2),(y,4)\}$	$\{(y,2)\}$
3	$\emptyset$	$\emptyset$
4	$\{(y,?),(y,2),(y,4)\}$	$\{(y,4)\}$
5	$\{(x,?),(x,1),(x,5)\}$	$\{(x,5)\}$

RD Example: Equations

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while $[x > 1]^3$ **do** (
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$$

$$
\begin{aligned}
\mathrm{RD}_{exit}(1) &= (\mathrm{RD}_{entry}(1) \backslash \{(x, ?), (x, 1), (x, 5)\}) \cup \{(x, 1)\}\\
\mathrm{RD}_{exit}(2) &= (\mathrm{RD}_{entry}(2) \backslash \{(y, ?), (y, 2), (y, 4)\}) \cup \{(y, 2)\}\\
\mathrm{RD}_{exit}(3) &= \mathrm{RD}_{entry}(3)\\
\mathrm{RD}_{exit}(4) &= (\mathrm{RD}_{entry}(4) \backslash \{(y, ?), (y, 2), (y, 4)\}) \cup \{(y, 4)\}\\
\mathrm{RD}_{exit}(5) &= (\mathrm{RD}_{entry}(5) \backslash \{(x, ?), (x, 1), (x, 5)\}) \cup \{(x, 5)\}
\end{aligned}
$$

RD Example: Equations

$[\ x := 5\ ]^1;$
$[\ y := 1\ ]^2;$
while $[x > 1]^3$ **do** (
$\quad [\ y := x * y\ ]^4;$
$\quad [\ x := x - 1\ ]^5$)

RD Example: Equations

$$
\begin{array}{l}
[\ x := 5\ ]^1; \\
[\ y := 1\ ]^2; \\
\textbf{while}\ [x > 1]^3\ \textbf{do}\ (\\
\qquad [\ y := x * y\ ]^4; \\
\qquad [\ x := x - 1\ ]^5\ )
\end{array}
$$

$$
\begin{aligned}
\mathrm{RD}_{entry}(1) &= \{(x, ?), (y, ?)\} \\
\mathrm{RD}_{entry}(2) &= \mathrm{RD}_{exit}(1) \\
\mathrm{RD}_{entry}(3) &= \mathrm{RD}_{exit}(2) \cup \mathrm{RD}_{exit}(5) \\
\mathrm{RD}_{entry}(4) &= \mathrm{RD}_{exit}(3) \\
\mathrm{RD}_{entry}(5) &= \mathrm{RD}_{exit}(4)
\end{aligned}
$$

RD Example: Equations

$$
\begin{aligned}
\mathrm{RD}_{entry}(1) &= \{(x,?),(y,?)\} \\
\mathrm{RD}_{entry}(2) &= \mathrm{RD}_{exit}(1) \\
\mathrm{RD}_{entry}(3) &= \mathrm{RD}_{exit}(2) \cup \mathrm{RD}_{exit}(5) \\
\mathrm{RD}_{entry}(4) &= \mathrm{RD}_{exit}(3) \\
\mathrm{RD}_{entry}(5) &= \mathrm{RD}_{exit}(4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{RD}_{exit}(1) &= (\mathrm{RD}_{entry}(1) \backslash \{(x,?),(x,1),(x,5)\}) \cup \{(x,1)\} \\
\mathrm{RD}_{exit}(2) &= (\mathrm{RD}_{entry}(2) \backslash \{(y,?),(y,2),(y,4)\}) \cup \{(y,2)\} \\
\mathrm{RD}_{exit}(3) &= \mathrm{RD}_{entry}(3) \\
\mathrm{RD}_{exit}(4) &= (\mathrm{RD}_{entry}(4) \backslash \{(y,?),(y,2),(y,4)\}) \cup \{(y,4)\} \\
\mathrm{RD}_{exit}(5) &= (\mathrm{RD}_{entry}(5) \backslash \{(x,?),(x,1),(x,5)\}) \cup \{(x,5)\}
\end{aligned}
$$

RD Example: Solutions

ℓ	$\mathrm{RD}_{entry}(\ell)$	$\mathrm{RD}_{exit}(\ell)$
1	$\{(x,?),(y,?)\}$	$\{(y,?),(x,1)\}$
2	$\{(y,?),(x,1)\}$	$\{(x,1),(y,2)\}$
3	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,2),(y,4),(x,5)\}$
4	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,4),(x,5)\}$
5	$\{(x,1),(y,4),(x,5)\}$	$\{(y,4),(x,5)\}$

RD Example: Solutions

ℓ	$\mathrm{RD}_{entry}(\ell)$	$\mathrm{RD}_{exit}(\ell)$
1	$\{(x,?),(y,?)\}$	$\{(y,?),(x,1)\}$
2	$\{(y,?),(x,1)\}$	$\{(x,1),(y,2)\}$
3	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,2),(y,4),(x,5)\}$
4	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,4),(x,5)\}$
5	$\{(x,1),(y,4),(x,5)\}$	$\{(y,4),(x,5)\}$

$[\ x := 5\ ]^1;$
$[\ y := 1\ ]^2;$
while $[x > 1]^3$ **do** (
$[\ y := x * y\ ]^4;$
$[\ x := x - 1\ ]^5$)

Very Busy Expression Analysis

An expression is *very busy* at the exit from a label if, no matter what path is taken from the label, the expression *must* (is guaranteed to) always be used before any of the variables occurring in it are redefined. The aim of the *Very Busy Expressions Analysis* is to determine:

> *For each program point, which expressions* must *(is guaranteed to) be very busy at exit from the point.*

Very Busy Expression Analysis

An expression is *very busy* at the exit from a label if, no matter what path is taken from the label, the expression *must* (is guaranteed to) always be used before any of the variables occurring in it are redefined. The aim of the *Very Busy Expressions Analysis* is to determine:

> *For each program point, which expressions* must *(is guaranteed to) be very busy at exit from the point.*

A possible optimisation based on this information is to evaluate the expression at the block and store its value for later use; this optimisation is sometimes called *hoisting* the expression.

Example

We illustrate this analysis with the following example:

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
\quad \qquad\quad\ \ [\ y := a - b\ ]^3\ ) \\
\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
\quad \qquad\quad\ \ [\ x := a - b\ ]^5\ )
\end{array}
$$

Example

We illustrate this analysis with the following example:

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
\qquad\qquad [\ y := a - b\ ]^3\ ) \\
\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
\qquad\qquad [\ x := a - b\ ]^5\ )
\end{array}
$$

The expressions $a - b$ and $b - a$ are both very busy at the start of the program (label 1). They can be hoisted resulting in a code size reduction.

VB Analysis

$$\mathit{kill}_{\mathsf{VB}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

VB Analysis

$$kill_{\mathsf{VB}} : \mathbf{Block}_\star \rightarrow \mathcal{P}(\mathbf{AExp}_\star)$$

$$gen_{\mathsf{VB}} : \mathbf{Block}_\star \rightarrow \mathcal{P}(\mathbf{AExp}_\star)$$

VB Analysis

$$kill_{\text{VB}} : \textbf{Block}_\star \to \mathcal{P}(\textbf{AExp}_\star)$$

$$gen_{\text{VB}} : \textbf{Block}_\star \to \mathcal{P}(\textbf{AExp}_\star)$$

$$\text{VB}_{entry} : \textbf{Lab}_\star \to \mathcal{P}(\textbf{AExp}_\star)$$

VB Analysis

$$kill_{\mathrm{VB}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$gen_{\mathrm{VB}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$\mathrm{VB}_{entry} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$\mathrm{VB}_{exit} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

VB Analysis

$$kill_{\mathrm{VB}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$gen_{\mathrm{VB}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$\mathrm{VB}_{entry} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

$$\mathrm{VB}_{exit} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{AExp}_\star)$$

The analysis is a *backward analysis* and we are interested in the *largest* sets satisfying the equation for VB_{exit}.

VB Auxiliary Functions

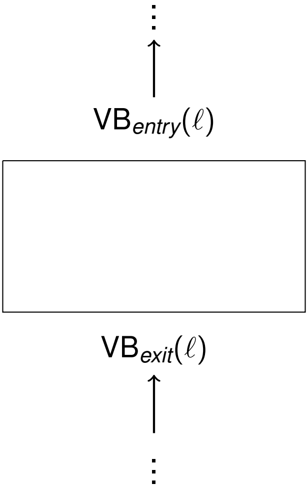
$$
\begin{aligned}
\mathit{kill}_{\mathrm{VB}}([\, x := a \,]^{\ell}) &= \{a' \in \mathbf{AExp}_{\star} \mid x \in \mathit{FV}(a')\} \\
\mathit{kill}_{\mathrm{VB}}([\, \mathbf{skip} \,]^{\ell}) &= \emptyset \\
\mathit{kill}_{\mathrm{VB}}([b]^{\ell}) &= \emptyset
\end{aligned}
$$

VB Auxiliary Functions

$$\begin{array}{rcl} kill_{\mathsf{VB}}([\, x := a\,]^{\ell}) & = & \{a' \in \mathbf{AExp}_{\star} \mid x \in FV(a')\} \\ kill_{\mathsf{VB}}([\, \mathbf{skip}\,]^{\ell}) & = & \emptyset \\ kill_{\mathsf{VB}}([b]^{\ell}) & = & \emptyset \end{array}$$

$$\begin{array}{rcl} gen_{\mathsf{VB}}([\, x := a\,]^{\ell}) & = & \mathbf{AExp}(a) \\ gen_{\mathsf{VB}}([\, \mathbf{skip}\,]^{\ell}) & = & \emptyset \\ gen_{\mathsf{VB}}([b]^{\ell}) & = & \mathbf{AExp}(b) \end{array}$$

VB Local Change



VB Local Change

$\vdots$

$\uparrow$

$\mathsf{VB}_{entry}(\ell)$

$[\ z := x + y\ ]^{\ell}\ \surd$

$\mathsf{VB}_{exit}(\ell)$

$\uparrow$

$\vdots$

VB Local Change

$$\vdots$$

$$\uparrow$$

$$\mathrm{VB}_{entry}(\ell)$$

$$[x + y < x]^{\ell} \quad \surd$$

$$\mathrm{VB}_{exit}(\ell)$$

$$\uparrow$$

$$\vdots$$

VB Local Change

$$\vdots$$

$$\uparrow$$

$$\mathrm{VB}_{entry}(\ell)$$

$$[\ x := x + y\ ]^{\ell} \ \surd$$

$$\mathrm{VB}_{exit}(\ell)$$

$$\uparrow$$

$$\vdots$$

VB Local Change

$$\vdots$$

$$\uparrow$$

$$\mathrm{VB}_{entry}(\ell)$$

$$\begin{array}{rl} [x + y < x]^\ell & \surd \\ [\ {\color{blue}z} := x + y\ ]^\ell & \surd \\ [\ {\color{red}x} := x + y\ ]^\ell & \surd \end{array}$$

$$\mathrm{VB}_{exit}(\ell)$$

$$\uparrow$$

$$\vdots$$

Whenever a variable x in an expression gets a new value it does not help us if it was evaluated before.

VB Equation Schemes

$$\mathsf{VB}_{exit}(\ell) \quad = \quad \begin{cases} \emptyset, \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcap\{\mathsf{VB}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

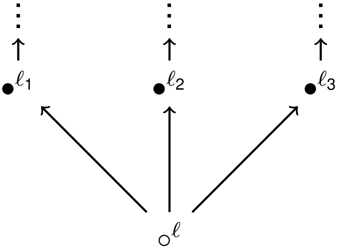
VB Equation Schemes

$$\mathsf{VB}_{exit}(\ell) = \begin{cases} \emptyset, \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcap\{\mathsf{VB}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

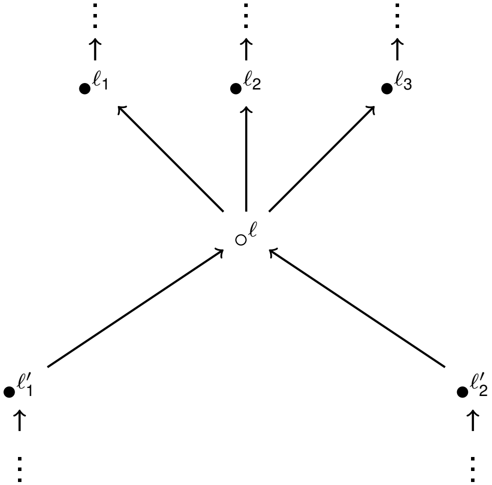
$$\mathsf{VB}_{entry}(\ell) = (\mathsf{VB}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{VB}}([B]^\ell)) \cup \mathit{gen}_{\mathsf{VB}}(B^\ell) \\ \text{where } [B]^\ell \in \mathit{blocks}(S_\star)$$

VB Global Collection

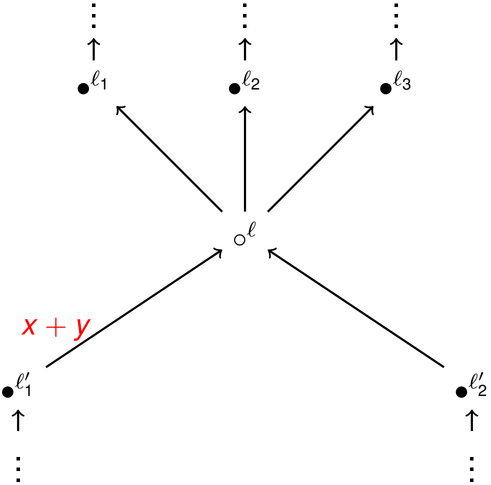
VB Global Collection



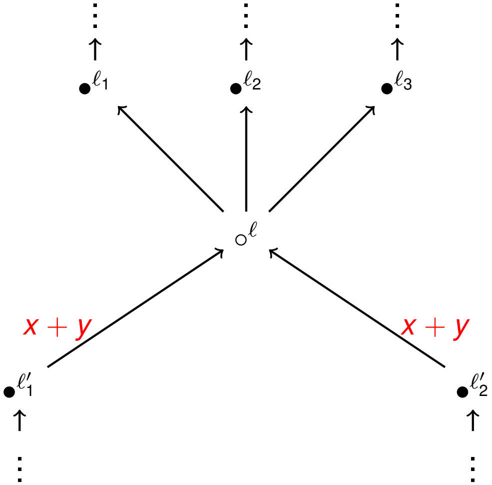
VB Global Collection



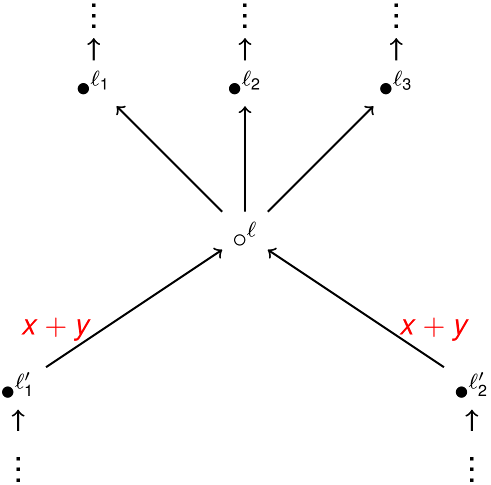
VB Global Collection



VB Global Collection



VB Global Collection



We need to go “back in time”.

VB Example

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
\qquad\qquad\ \ [\ y := a - b\ ]^3\ ) \\
\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
\qquad\qquad\ \ [\ x := a - b\ ]^5\ )
\end{array}
$$

VB Example

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
\qquad\qquad [\ y := a - b\ ]^3\ ) \\
\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
\qquad\qquad [\ x := a - b\ ]^5\ )
\end{array}
$$

ℓ	$kill_{\mathrm{VB}}(\ell)$	$gen_{\mathrm{VB}}(\ell)$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	$\{b - a\}$
3	$\emptyset$	$\{a - b\}$
4	$\emptyset$	$\{b - a\}$
5	$\emptyset$	$\{a - b\}$

VB Example: Equations

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\,[\, x := b - a \,]^2; \\
\qquad\qquad\; [\, y := a - b \,]^3 \,) \\
\quad \textbf{else } (\,[\, y := b - a \,]^4; \\
\qquad\qquad\; [\, x := a - b \,]^5 \,)
\end{array}
$$

VB Example: Equations

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
\qquad\qquad [\ y := a - b\ ]^3\ ) \\
\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
\qquad\qquad [\ x := a - b\ ]^5\ )
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{VB}_{entry}(1) & = & \mathrm{VB}_{exit}(1) \\
\mathrm{VB}_{entry}(2) & = & \mathrm{VB}_{exit}(2) \cup \{b - a\} \\
\mathrm{VB}_{entry}(3) & = & \{a - b\} \\
\mathrm{VB}_{entry}(4) & = & \mathrm{VB}_{exit}(4) \cup \{b - a\} \\
\mathrm{VB}_{entry}(5) & = & \{a - b\}
\end{array}
$$

VB Example: Equations

$$
\begin{aligned}
&\textbf{if } [a > b]^1 \\
&\quad \textbf{then } (\ [\ x := b - a\ ]^2; \\
&\qquad\qquad [\ y := a - b\ ]^3\ ) \\
&\quad \textbf{else } (\ [\ y := b - a\ ]^4; \\
&\qquad\qquad [\ x := a - b\ ]^5\ )
\end{aligned}
$$

VB Example: Equations

$$
\begin{array}{l}
\textbf{if } [a > b]^1 \\
\quad \textbf{then } (\, [\, x := b - a \,]^2; \\
\qquad\qquad [\, y := a - b \,]^3 \,) \\
\quad \textbf{else } (\, [\, y := b - a \,]^4; \\
\qquad\qquad [\, x := a - b \,]^5 \,)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{VB}_{exit}(1) & = & \mathrm{VB}_{entry}(2) \cap \mathrm{VB}_{entry}(4) \\
\mathrm{VB}_{exit}(2) & = & \mathrm{VB}_{entry}(3) \\
\mathrm{VB}_{exit}(3) & = & \emptyset \\
\mathrm{VB}_{exit}(4) & = & \mathrm{VB}_{entry}(5) \\
\mathrm{VB}_{exit}(5) & = & \emptyset
\end{array}
$$

VB Example: Equations

$$
\begin{aligned}
\mathrm{VB}_{entry}(1) &= \mathrm{VB}_{exit}(1) \\
\mathrm{VB}_{entry}(2) &= \mathrm{VB}_{exit}(2) \cup \{b - a\} \\
\mathrm{VB}_{entry}(3) &= \{a - b\} \\
\mathrm{VB}_{entry}(4) &= \mathrm{VB}_{exit}(4) \cup \{b - a\} \\
\mathrm{VB}_{entry}(5) &= \{a - b\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VB}_{exit}(1) &= \mathrm{VB}_{entry}(2) \cap \mathrm{VB}_{entry}(4) \\
\mathrm{VB}_{exit}(2) &= \mathrm{VB}_{entry}(3) \\
\mathrm{VB}_{exit}(3) &= \emptyset \\
\mathrm{VB}_{exit}(4) &= \mathrm{VB}_{entry}(5) \\
\mathrm{VB}_{exit}(5) &= \emptyset
\end{aligned}
$$

VB Example: Solutions

ℓ	$\mathrm{VB}_{entry}(\ell)$	$\mathrm{VB}_{exit}(\ell)$
1	$\{a - b, b - a\}$	$\{a - b, b - a\}$
2	$\{a - b, b - a\}$	$\{a - b\}$
3	$\{a - b\}$	$\emptyset$
4	$\{a - b, b - a\}$	$\{a - b\}$
5	$\{a - b\}$	$\emptyset$

VB Example: Solutions

ℓ	$\mathrm{VB}_{entry}(\ell)$	$\mathrm{VB}_{exit}(\ell)$
1	$\{a - b, b - a\}$	$\{a - b, b - a\}$
2	$\{a - b, b - a\}$	$\{a - b\}$
3	$\{a - b\}$	$\emptyset$
4	$\{a - b, b - a\}$	$\{a - b\}$
5	$\{a - b\}$	$\emptyset$

if $[a > b]^1$
  then ($[\ x := b - a\ ]^2$;
    $[\ y := a - b\ ]^3$)
  else ($[\ y := b - a\ ]^4$;
    $[\ x := a - b\ ]^5$)

Live Variable Analysis

A variable is *live* at the exit from a label if there exists a path from the label to a use of the variable that does not re-define the variable. The *Live Variables Analysis* will determine:

> *For each program point, which variables* may *be live at the exit from the point.*

Live Variable Analysis

A variable is *live* at the exit from a label if there exists a path from the label to a use of the variable that does not re-define the variable. The *Live Variables Analysis* will determine:

> *For each program point, which variables* may *be live at the exit from the point.*

This analysis might be used as the basis for *Dead Code Elimination*. If the variable is not live at the exit from a label then, if the elementary block is an assignment to the variable, the elementary block can be eliminated.

Example

The example program to illustrate the *LV* analysis is:

$$
\begin{array}{l}
[\, x := 2 \,]^1; \\
[\, y := 4 \,]^2; \\
[\, x := 1 \,]^3; \\
(\, \textbf{if } [y > x]^4 \\
\quad \textbf{then } [\, z := y \,]^5 \\
\quad \textbf{else } [\, z := y * y \,]^6 \,); \\
[\, x := z \,]^7
\end{array}
$$

Example

The example program to illustrate the *LV* analysis is:

$$
\begin{array}{l}
[\, x := 2 \,]^1; \\
[\, y := 4 \,]^2; \\
[\, x := 1 \,]^3; \\
(\ \textbf{if}\ [y > x]^4 \\
\quad \textbf{then}\ [\, z := y \,]^5 \\
\quad \textbf{else}\ [\, z := y * y \,]^6\ ); \\
[\, x := z \,]^7
\end{array}
$$

The variable x is not live at the exit from 1; the first assignment to x is thus redundant and can be eliminated. Both x and y are alive at the exit from label 3.

LV Analysis

$$kill_{\mathsf{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

LV Analysis

$$kill_{\mathsf{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$gen_{\mathsf{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

LV Analysis

$$\mathit{kill}_{\mathrm{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathit{gen}_{\mathrm{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathrm{LV}_{\mathit{entry}} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

LV Analysis

$$kill_{\mathsf{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$gen_{\mathsf{LV}} : \mathbf{Block}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathsf{LV}_{entry} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathsf{LV}_{exit} : \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Var}_\star)$$

LV Analysis

$$kill_{\mathrm{LV}} : \mathbf{Block}_\star \rightarrow \mathcal{P}(\mathbf{Var}_\star)$$

$$gen_{\mathrm{LV}} : \mathbf{Block}_\star \rightarrow \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathrm{LV}_{entry} : \mathbf{Lab}_\star \rightarrow \mathcal{P}(\mathbf{Var}_\star)$$

$$\mathrm{LV}_{exit} : \mathbf{Lab}_\star \rightarrow \mathcal{P}(\mathbf{Var}_\star)$$

The analysis is a *backward analysis* and we are interested in the *smallest* sets satisfying the equation for LV_{exit}.

LV Auxiliary Functions

$$
\begin{aligned}
\mathit{kill}_{\mathrm{LV}}([\, x := a \,]^{\ell}) &= \{x\} \\
\mathit{kill}_{\mathrm{LV}}([\, \textbf{skip} \,]^{\ell}) &= \emptyset \\
\mathit{kill}_{\mathrm{LV}}([b]^{\ell}) &= \emptyset
\end{aligned}
$$

LV Auxiliary Functions

$$
\begin{aligned}
\mathit{kill}_{\mathrm{LV}}([\,x := a\,]^{\ell}) &= \{x\} \\
\mathit{kill}_{\mathrm{LV}}([\,\mathbf{skip}\,]^{\ell}) &= \emptyset \\
\mathit{kill}_{\mathrm{LV}}([b]^{\ell}) &= \emptyset
\end{aligned}
$$

$$
\begin{aligned}
\mathit{gen}_{\mathrm{LV}}([\,x := a\,]^{\ell}) &= \mathit{FV}(a) \\
\mathit{gen}_{\mathrm{LV}}([\,\mathbf{skip}\,]^{\ell}) &= \emptyset \\
\mathit{gen}_{\mathrm{LV}}([b]^{\ell}) &= \mathit{FV}(b)
\end{aligned}
$$

LV Equation Schemes

$$\mathsf{LV}_{\mathit{exit}}(\ell) = \begin{cases} \emptyset, \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{\mathit{entry}}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

LV Equation Schemes

$$\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset, \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) = (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}([B]^\ell) \cup \mathit{gen}_{\mathsf{LV}}([B]^\ell)$$
$$\text{where } [B]^\ell \in \mathit{blocks}(S_\star)$$

LV Example

$$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$$
$$(\textbf{if}\ \ [y > x]^4\ \textbf{then}\ \ [\ z := y\ ]^5\ \textbf{else}\ \ [\ z := y * y\ ]^6\ );$$
$$[\ x := z\ ]^7$$

LV Example

$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$
$(\textbf{if}\ \ [y > x]^4\ \textbf{then}\ \ [\ z := y\ ]^5\ \textbf{else}\ \ [\ z := y * y\ ]^6\ );$
$[\ x := z\ ]^7$

ℓ	$kill_{\mathrm{LV}}(\ell)$	$gen_{\mathrm{LV}}(\ell)$
1	$\{x\}$	$\emptyset$
2	$\{y\}$	$\emptyset$
3	$\{x\}$	$\emptyset$
4	$\emptyset$	$\{x, y\}$
5	$\{z\}$	$\{y\}$
6	$\{z\}$	$\{y\}$
7	$\{x\}$	$\{z\}$

LV Example: Equations

$$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$$
$$(\textbf{if}\ [y > x]^4\ \textbf{then}\ [\ z := y\ ]^5\ \textbf{else}\ [\ z := y * y\ ]^6\ );$$
$$[\ x := z\ ]^7$$

LV Example: Equations

$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$
$(\textbf{if}\ [y > x]^4\ \textbf{then}\ [\ z := y\ ]^5\ \textbf{else}\ [\ z := y * y\ ]^6\ );$
$[\ x := z\ ]^7$

$$
\begin{aligned}
LV_{entry}(1) &= LV_{exit}(1) \setminus \{x\} \\
LV_{entry}(2) &= LV_{exit}(2) \setminus \{y\} \\
LV_{entry}(3) &= LV_{exit}(3) \setminus \{x\} \\
LV_{entry}(4) &= LV_{exit}(4) \cup \{x, y\} \\
LV_{entry}(5) &= (LV_{exit}(5) \setminus \{z\}) \cup \{y\} \\
LV_{entry}(6) &= (LV_{exit}(6) \setminus \{z\}) \cup \{y\} \\
LV_{entry}(7) &= \{z\}
\end{aligned}
$$

LV Example: Equations

$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$
$(\textbf{if}\ \ [y > x]^4\ \textbf{then}\ \ [\ z := y\ ]^5\ \textbf{else}\ \ [\ z := y * y\ ]^6\ );$
$[\ x := z\ ]^7$

LV Example: Equations

$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$
$(\textbf{if}\ \ [y > x]^4\ \textbf{then}\ \ [\ z := y\ ]^5\ \textbf{else}\ \ [\ z := y * y\ ]^6\ );$
$[\ x := z\ ]^7$

$$
\begin{aligned}
\mathrm{LV}_{exit}(1) &= \mathrm{LV}_{entry}(2)\\
\mathrm{LV}_{exit}(2) &= \mathrm{LV}_{entry}(3)\\
\mathrm{LV}_{exit}(3) &= \mathrm{LV}_{entry}(4)\\
\mathrm{LV}_{exit}(4) &= \mathrm{LV}_{entry}(5) \cup \mathrm{LV}_{entry}(6)\\
\mathrm{LV}_{exit}(5) &= \mathrm{LV}_{entry}(7)\\
\mathrm{LV}_{exit}(6) &= \mathrm{LV}_{entry}(7)\\
\mathrm{LV}_{exit}(7) &= \emptyset
\end{aligned}
$$

LV Example: Solutions

ℓ	$\mathsf{LV}_{entry}(\ell)$	$\mathsf{LV}_{exit}(\ell)$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	$\{y\}$
3	$\{y\}$	$\{x, y\}$
4	$\{x, y\}$	$\{y\}$
5	$\{y\}$	$\{z\}$
6	$\{y\}$	$\{z\}$
7	$\{z\}$	$\emptyset$

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6	$\{y\}$	$\{z\}$
7	$\{z\}$	$\emptyset$

$[\ x := 2\ ]^1;\ [\ y := 4\ ]^2;\ [\ x := 1\ ]^3;$
$(\textbf{if}\ \ [y > x]^4\ \textbf{then}\ \ [\ z := y\ ]^5\ \textbf{else}\ \ [\ z := y * y\ ]^6\ );$
$[\ x := z\ ]^7$

LV Variations

Some authors assume that the variables of interest are output at the end of the program.

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In that case $\mathrm{LV}_{exit}(7)$ should be $\{x, y, z\}$ which means that $\mathrm{LV}_{entry}(7)$, $\mathrm{LV}_{exit}(5)$ and $\mathrm{LV}_{exit}(6)$ should all be $\{y, z\}$.