Robust Optimisation & its Guarantees

Berç Rustem

Imperial College London

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with D. Kuhn, P. Parpas, W. Wiesemann,
R. Fonseca, M. Kapsos, S.Žaković, S. Zymler

Outline

Outline

Introduction to Robust Optimisation

- Generic problem:

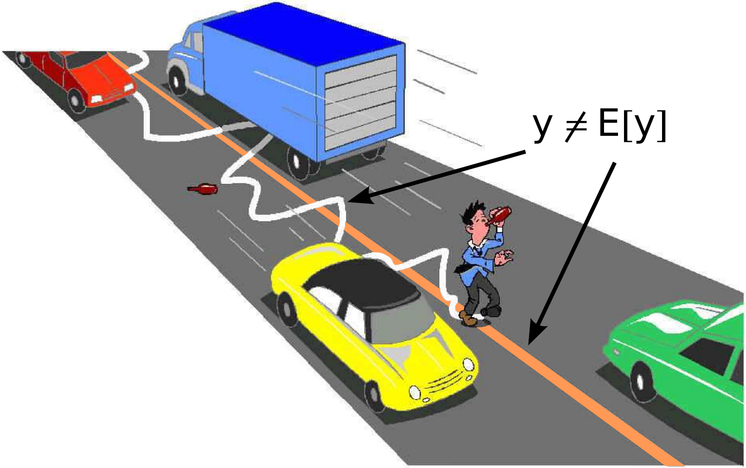
$$\min_{\mathbf{x}\in X} f(\mathbf{x}, \mathbf{y})$$

  - $\mathbf{x} \in X$: decision variables
  - $\mathbf{y}$: problem specific data

- Uncertainty in $\mathbf{y}$ due to:
  - Inaccurate forecasts
  - Inaccurate assumptions (e.g. distributions)
  - etc.

- Disregarding uncertainty $\implies$ bad decisions ...

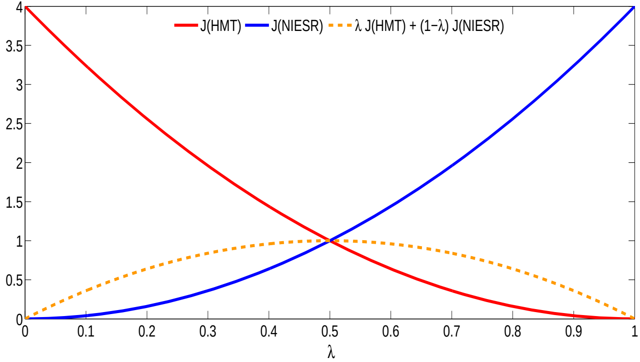
Deterministic Optimisation



$$\text{alive}\left(\mathbb{E}\left[\text{position}\right]\right) = \text{true}, \quad \text{but} \quad \mathbb{E}\left[\text{alive}\left(\text{position}\right)\right] = \text{false!}$$

Example: Macroeconomic Policy with Rival Models

$$\underset{x}{\text{minimise}} \ \{\lambda \ J_{\text{HMT}}(Y_{\text{HMT}}(x), x) + (1-\lambda) \ J_{\text{NIESR}}(Y_{\text{NIESR}}(x), x)\}$$ [1,2]



[1] Becker et al. [1986]
[2] R et al. [2000]

Uncertainty Set

- Set $\mathcal{U}$ for $\mathbf{y}$, $\mathbf{y} \in \mathcal{U}$ with high confidence.
- Typical $\mathcal{U}$:
  - Discrete: $\mathcal{U} = \{\hat{\mathbf{y}}_0, \ldots, \hat{\mathbf{y}}_i, \ldots, \hat{\mathbf{y}}_k\}, i \in I$.
  - Interval: $\mathcal{U} = \{\mathbf{y} \; : \; \underline{\mathbf{y}} \leq \mathbf{y} \leq \overline{\mathbf{y}}\}$.
  - Ellipsoid: $\mathcal{U} = \{\mathbf{y} \; : \; \|\mathbf{A}\mathbf{y}\|_2 \leq \delta\}$
- Robust – worst-case – Optimisation: best decision $\mathbf{x} \in \mathcal{X}$ in view of worst possible scenario $\mathbf{y} \in \mathcal{U}$
- Minimax problem:

$$\min_{\mathbf{x} \in X} \max_{\mathbf{y} \in \mathcal{U}} f(\mathbf{x}, \mathbf{y}).$$

Discrete Minimax

$$\min_{x \in X} \max_{\lambda} \left\{ \sum_i \lambda^i J^i(x) \;\middle|\; \sum_i \lambda^i = 1, \lambda^i \geq 0, \forall i \right\}$$

$$\Longleftrightarrow$$

$$\min_{x \in X} \;\; \max_{i \in \mathcal{U}} \left\{ J^i(x) \right\}^3$$

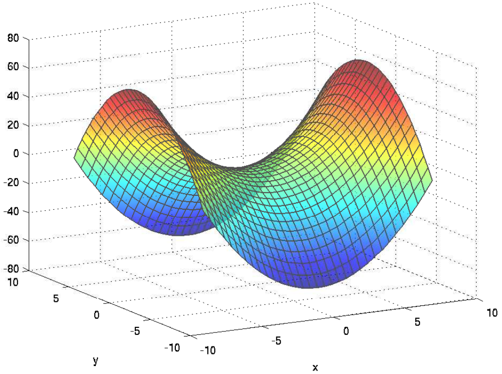
$$\Longleftrightarrow$$

$$\min_{x \in X, v \in \mathbb{R}^1} \left\{ v \mid v \geq J^i(x), i \in \mathcal{U} \right\}$$

[3] Obasanjo et al. [2010]

Saddlepoint Solution

- $f(\mathbf{x}, \mathbf{y})$ convex in $\mathbf{x}$ & concave in $\mathbf{y}$
- minimax $\Rightarrow$ saddlepoint
- elegant models & powerful algorithms



$$f(\mathbf{x}^*, \mathbf{y}) \leq f(\mathbf{x}^*, \mathbf{y}^*) \leq f(\mathbf{x}, \mathbf{y}^*) \quad \forall \mathbf{x} \in X, \mathbf{y} \in \mathcal{U}.$$

LP Duality

- For every Primal a Dual can be constructed:
  - Primal:

$$\begin{aligned} &\text{minimise} \quad \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{P}$$

  - Dual:

$$\begin{aligned} &\text{maximise} \quad \boldsymbol{b}^T \boldsymbol{y} \\ &\text{subject to} \quad \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0} \end{aligned} \tag{D}$$

- (P) feasible:

$$\exists \hat{\boldsymbol{x}} \in \mathbb{R}^n \quad \text{such that} \quad \boldsymbol{A}\hat{\boldsymbol{x}} \leq \boldsymbol{b}, \quad \hat{\boldsymbol{x}} \geq \boldsymbol{0}$$

$\Longrightarrow$

$$\text{Opt}(P) = \text{Opt}(D) \quad (\text{Strong Duality})$$

Dualising minimax

Original: $$\min_{x \geq 0: Ax \geq b} \underbrace{\max_{y \geq 0: Wy \leq h} c^T x + d^T y + x^T Q y}$$

Inner: $$\underbrace{\max_{y \geq 0: Wy \leq h} c^T x + (Qx + d)^T y}$$

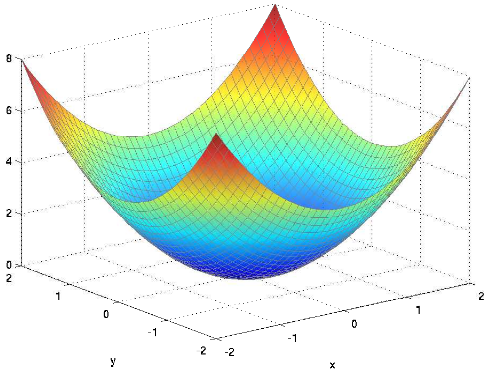
Inner dual: $$\begin{array}{ll} \min & h^T \lambda + c^T x \\ \text{s.t.} & \lambda \geq 0, W^T \lambda \geq Qx + d \end{array}$$

Original equivalent:

$$\min_{x \geq 0: Ax \geq b} \left\{ c^T x + h^T \lambda \,\middle|\, \lambda \geq 0, W^T \lambda \geq Qx + d \right\}$$

Multiple Maxima

- Global optimisation:
- Generally, multiple (global) maxima for $\mathbf{y}$:



$$f(x,y) = x^2 + y^2 \quad f(x^*,y^*) = 4 \quad x^* = 0,\ y^* = 2 \vee -2$$

Outline

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Mean-Variance Portfolio Optimisation

Optimal Asset Allocation

Compute $\boldsymbol{w} \in \mathbb{R}^n$ for high return & low risk $\rho(\boldsymbol{w})$

- Mean-Variance Portfolio Optimisation:

$$\max_{\boldsymbol{w} \in \mathbb{R}^n} \left\{ \boldsymbol{w}^T \boldsymbol{\mu} - \lambda \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \;\middle|\; \boldsymbol{w} \in \mathcal{W} \right\}$$

  - Expected return: $\boldsymbol{w}^T \boldsymbol{\mu}$
  - Risk: $\rho(\boldsymbol{w}) \equiv \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$
  - Risk aversion: λ
    $\Leftrightarrow \lambda : \max_{\boldsymbol{w} \in \mathbb{R}^n} \{ \boldsymbol{w}^T \boldsymbol{\mu} \mid \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \leq v; \boldsymbol{w} \in \mathcal{W} \}$

Robust Mean-Variance Portfolio Optimisation

Robust Risk **and** Return [4]

$\boldsymbol{w}$: optimal risk-return

$\boldsymbol{\Gamma}, \boldsymbol{r}$: worst-case

- Worst-case optimal $\boldsymbol{w}, \boldsymbol{r}, \boldsymbol{\Gamma}$:

$$\begin{aligned} \min_{\boldsymbol{w}\in\mathbb{R}^n} \quad \max_{\boldsymbol{\Gamma}\in\mathcal{U}_{\boldsymbol{\Gamma}},\, \boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \quad & \boldsymbol{w}^T\Sigma(\boldsymbol{\Gamma}, \boldsymbol{r})\boldsymbol{w} = \boldsymbol{w}^T(\boldsymbol{\Gamma} - \boldsymbol{r}\boldsymbol{r}^T)\boldsymbol{w} \\ \text{s.t} \quad & \min_{\boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T\boldsymbol{r} \geq R, \\ & \boldsymbol{w} \in \mathcal{W}. \end{aligned}$$

- $\boldsymbol{\Gamma}$: Second moment;
- $\boldsymbol{\Gamma} - \boldsymbol{r}\boldsymbol{r}^T \succeq 0$
- Consistent mean & covariance

[4] Ye et al. [2012]

Outline

Minimum risk & rival covariances

[Kapsos & R: 2014]

- Covariances unknown!
- Estimated with error .
- Robust statistics: shrinkage to reduce estimation error.

Example: *Shrinkage*: Averaging sample & structured estimator

Step 1

$$\Sigma^S = \delta^* \hat{\Sigma}^1 + (1 - \delta^*)\hat{\Sigma}^2$$
$$\delta^* = arg \min_{\delta} E(||\delta\hat{\Sigma}^1 + (1-\delta)\hat{\Sigma}^2 - \Sigma||^2)$$

Step 2

$$\min_{\boldsymbol{w} \in \mathcal{W}} \{\boldsymbol{w}^T \Sigma^S \boldsymbol{w}\}$$

Simultaneously: best $\boldsymbol{w}$ & worst case Σ

Estimator $\Sigma^S = \sum_{i=1}^m \delta_i \hat{\Sigma}^i;\ \delta \in \{[0,1], \sum_i \delta_i = 1\}$.

Robust model[5]

$$\min_{\boldsymbol{w}\in\mathcal{W}} \max_{\delta} \boldsymbol{w}^\top \Sigma^S(\delta)\boldsymbol{w}$$

$$\min_{\boldsymbol{w}\in\mathcal{W}} \max_{\delta} \sum_i \delta_i \boldsymbol{w}^\top \hat{\Sigma}^i \boldsymbol{w}$$

$$\min_{\boldsymbol{w}\in\mathcal{W}} \max_{i} \boldsymbol{w}^\top \hat{\Sigma}^i \boldsymbol{w}.$$

$\Leftrightarrow$

$$\min_{\theta\in\mathbb{R},\boldsymbol{w}\in\mathcal{W}} \theta$$

$$\text{s.t. } \boldsymbol{w}^\top \hat{\Sigma}^i \boldsymbol{w} \leq \theta, \qquad \forall\ i = 1, ..., m.$$

[5] Kapsos and R [2014]

Equally-weighted Risk Contribution

[Kapsos, Christofides, Parpas & R: 2012]

- Assets contribute equal risk
- Risk measure: variance
- $\Leftrightarrow$ minimum var + diversification

- $\frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{w}_i} = \frac{\partial \rho(\boldsymbol{w})}{\partial \boldsymbol{w}_j}, \quad \forall i, j$
- $\rho(\boldsymbol{w}) = \boldsymbol{w}^T \Sigma \boldsymbol{w}$
- $\min_{\boldsymbol{w}} \left\{ \boldsymbol{w}^T \Sigma \boldsymbol{w} \,|\, \sum_i \ln \boldsymbol{w}_i \geq c \right\}$

In real life...

- Σ unknown
- Estimation error.
- Time-varying

Robust Model

$$\min_{\boldsymbol{w} \in \mathcal{W}} \max_{\Sigma \in \mathcal{S}} \left\{ \boldsymbol{w}^T \Sigma \boldsymbol{w} \,\middle|\, \sum_i \ln w_i \geq c \right\}$$

Discrete or continuous uncertainty sets for Σ: [6]

[6] Kapsos et al. [2012]

Robust Equally-weighted Risk Contribution

Discrete uncertainty Convex

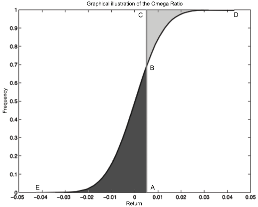
$$\min \theta$$
$$\text{s.t.} \quad \sum_{i=1}^{n} \ln w_i \geq c$$
$$\boldsymbol{w}^T \Sigma^j \boldsymbol{w} \leq \theta, \; j = 1, ..., m$$
$$\boldsymbol{w} \geq 0$$
$$\mathcal{S} = \{\Sigma^j\}, \; j = 1, ..., m.$$

Continuous uncertainty SIP

$$\min_{\boldsymbol{w}} \max_{\Sigma} \boldsymbol{w}^T \Sigma \boldsymbol{w}$$
$$\text{s.t.} \quad \sum_{i=1}^{n} \ln w_i \geq c$$
$$\Sigma^l \leq \Sigma \leq \Sigma^u$$
$$\boldsymbol{w} \geq 0$$
$$\Sigma \succcurlyeq 0.$$

Ω ratio

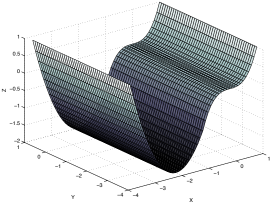
[Kapsos, Zymler, Christofides & R: 2011]



- Performance measure for non-normal return distribution
- $\Omega \text{ ratio} = \dfrac{\textit{Light Grey Area}}{\textit{Dark Grey Area}}$

Maximising Ω ratio

$$\max_{\boldsymbol{w}} \left\{ \frac{\int_{\tau}^{+\infty}[1 - F(\boldsymbol{w}^\top y)]\mathrm{d}y}{\int_{-\infty}^{\tau} F(\boldsymbol{w}^\top y)\mathrm{d}y} \,\middle|\, \boldsymbol{w} \in \mathcal{W} \right\}$$



- Quasi-convex problem
- Solution: family of convex problems or fractional LP

Ω ratio maximisation

Family of convex problems - continuous distributions[7]

$$\max_{\boldsymbol{w}} \left\{ \delta(\boldsymbol{w}^\top E_p(\boldsymbol{r}) - \tau) - (1-\delta) E_p([\tau - \boldsymbol{w}^\top \boldsymbol{r}]^+) \mid \boldsymbol{w} \in \mathcal{W} \right\}$$

solve for varying δ - keep solution with max Ω ratio

Fractional LP - discrete distributions

$$\max_{\boldsymbol{w}} \left\{ \frac{\boldsymbol{w}^T \bar{\boldsymbol{r}} - \tau}{\sum_j [\tau - \boldsymbol{w}^T \boldsymbol{r}_j]^+ p_j} \middle| \sum_i w_i = 1;\ \underline{\boldsymbol{w}} \leq \boldsymbol{w} \leq \bar{\boldsymbol{w}} \right\}$$

[7] Kapsos et al. [2011]

Worst-case Ω ratio

[Kapsos, Christofides & R: 2014]

Definition

Worst-case Ω (WCΩR) - fixed $\boldsymbol{w} \in \mathcal{W}$ - wrt set of probability distributions $\mathcal{P}$ or Π[8]:

$$WC\Omega R(\boldsymbol{w}) \equiv \inf_{p \in \mathcal{P}} \frac{\boldsymbol{w}^\top E_p(\boldsymbol{r}) - \tau}{E_p([\tau - \boldsymbol{w}^\top \boldsymbol{r}]^+)},$$

Discrete analogue

$$WC\Omega R(\boldsymbol{w}) \equiv \inf_{\boldsymbol{\pi} \in \Pi} \frac{\boldsymbol{w}^\top (\boldsymbol{R}^\top \boldsymbol{\pi}) - \tau}{\boldsymbol{\pi}^\top \left[\tau \mathbf{1} - (\boldsymbol{R}\boldsymbol{w})\right]^+}.$$

Density functions only known to belong to set $\mathcal{P}$ or Π.

[8] Kapsos et al. [2014]

Mixture distribution uncertainty

$$p(\boldsymbol{r}) \in \mathcal{P} = \left\{ \sum_{i=1}^{I} \lambda_i p^i(\boldsymbol{r}) : \lambda_i \in \Lambda \right\},$$

λ_i: unknown mixture weight for probability distribution $p^i(\boldsymbol{r})$.

$$\max_{\boldsymbol{w} \in \mathcal{W}, \theta \in \mathbb{R}} \theta$$

$$\text{s.t.}\ \delta \left(\boldsymbol{w}^T E_{p^i}(\boldsymbol{r}) - \tau \right) - (1-\delta)\, E_{p^i}([\tau - \boldsymbol{w}^T \boldsymbol{r}]^+) \geq \theta$$

$$\forall i = 1, ..., I.$$

Two distributions with same mean & variance

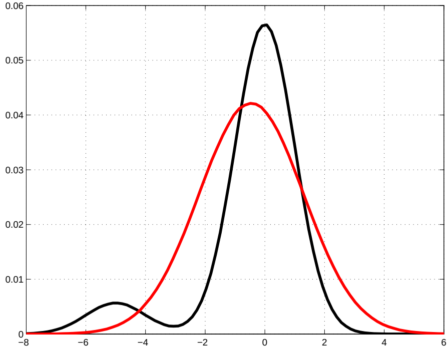


Figure: Two distributions with same mean and variance. The dotted distribution is a symmetric normal distribution. The dark line shows a negatively skewed distribution with fat tails. The Sharpe ratio is indifferent between the two. A rational investor will always prefer the red distribution.

Outline

Robust Portfolio Optimisation

[Zymler, Kuhn & R: 2010]

- $\tilde{\boldsymbol{r}}$: Asset returns.
- Portfolio return: $\boldsymbol{w}^T \tilde{\boldsymbol{r}}$.
- Max return[9,10]: $\max_{\boldsymbol{w} \in \mathcal{W}} \ \boldsymbol{w}^T \tilde{\boldsymbol{r}}$
- $\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}} \equiv \{ \boldsymbol{r} \mid (\boldsymbol{r} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \leq \delta^2 \}$
- Robust optimisation – worst-case:

$$\max_{\boldsymbol{w} \in \mathcal{W}} \min_{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}} \ \boldsymbol{w}^T \boldsymbol{r} \equiv \max_{\boldsymbol{w} \in \mathcal{W}} \ \boldsymbol{w}^T \boldsymbol{\mu} - \delta \left\| \boldsymbol{\Sigma}^{1/2} \boldsymbol{w} \right\|_2 .$$

[9] Ben-Tal and Nemirovski [1999]
[10] R and Howe [2002]

Guarantees

- Known means $\boldsymbol{\mu}$ & $\boldsymbol{\Sigma} \succ \mathbf{0}$, $\tilde{\boldsymbol{r}}$, but not entire distribution.
- $\mathcal{P}$ set of all distributions with mean $\boldsymbol{\mu}$ & cov $\boldsymbol{\Sigma}$.
- For any $\boldsymbol{w} \in \mathcal{W}, p$ & $\forall$ distributions $\in \mathcal{P}$

$$\delta = \sqrt{p/(1-p)} \implies \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\{\boldsymbol{w}^T\tilde{\boldsymbol{r}} \geq \min_{\boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T\boldsymbol{r}\} = p$$

with probability p better return than worst-case[11].

- Non-inferiority insurance:

$$\boxed{\theta^* = \max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T\boldsymbol{r}} \implies \boldsymbol{w}^{*T}\boldsymbol{r} \geq \theta^* \quad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}.$$

[11] El Ghaoui et al. [2003]

Support Information

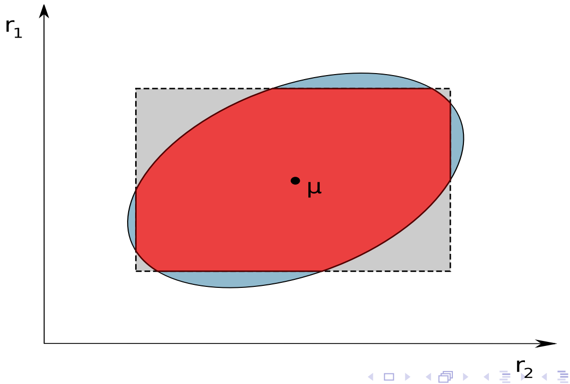
- Support on $\tilde{\boldsymbol{r}}$:

$$\mathcal{B} = \{\boldsymbol{r} : \boldsymbol{l} \leq \boldsymbol{r} \leq \boldsymbol{u}\} \quad (\text{or: } \mathcal{B} = \{\boldsymbol{r} : \boldsymbol{r} \geq \boldsymbol{0}\})$$

$\boldsymbol{r}$: realization of $\tilde{\boldsymbol{r}}$.

- Support with $\mathcal{U}_{\boldsymbol{r}}$:

$$\mathcal{U}_{\boldsymbol{r}} = \{\boldsymbol{r} \in \mathcal{B} \mid (\boldsymbol{r} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \leq \delta^2\}$$



Duality

- Strong convex duality:

$$\max_{\boldsymbol{w}\in\mathcal{W}} \min_{\boldsymbol{r}\in\mathcal{U}_{\boldsymbol{r}}} \boldsymbol{w}^T\boldsymbol{r} \equiv \max_{\boldsymbol{w}\in\mathcal{W},\boldsymbol{s}\geq\boldsymbol{0}} \boldsymbol{\mu}^T(\boldsymbol{w}-\boldsymbol{s})-\delta\left\|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{w}-\boldsymbol{s})\right\|_2.$$

  $\boldsymbol{s}$: dual variable.

- Consider ρ:

$$\rho(\boldsymbol{w}) = \min_{\boldsymbol{s}\geq\boldsymbol{0}} -\boldsymbol{\mu}^T(\boldsymbol{w}-\boldsymbol{s})+\delta\left\|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{w}-\boldsymbol{s})\right\|_2.$$

  ρ coherent risk-measure

- max worst-case return $\iff$ min coherent risk!

Modelling Option Returns

- Option weights: $\boldsymbol{w^d}$ & Returns $\tilde{\boldsymbol{r}}^{\boldsymbol{d}} \equiv f(\tilde{\boldsymbol{r}})$
- Call j strike K_j & call price C_j on underlying i, price S_0^i:

$$\tilde{r}_j^d = f_j(\tilde{\boldsymbol{r}}) = \frac{\max\left\{0, S_0^i \tilde{r}_i - K_j\right\}}{C_j}$$
$$= \max\left\{0, a_j + b_j \tilde{r}_i\right\}; a_j = -\frac{K_j}{C_j}, b_j = \frac{S_0^i}{C_j}.$$

- Put j with premium P_j:

$$\tilde{r}_j^d = f_j(\tilde{\boldsymbol{r}}) = \max\left\{0, a_j + b_j \tilde{r}_i\right\}; a_j = \frac{K_j}{P_j}, b_j = -\frac{S_0^i}{P_j}.$$

General form:

$$\boxed{\tilde{\boldsymbol{r}}^{\boldsymbol{d}} = f(\tilde{\boldsymbol{r}}) = \max\{0, \boldsymbol{a} + \boldsymbol{B}\tilde{\boldsymbol{r}}\}}$$

Incorporating Options in Robust Framework

- Portfolio return $\tilde{r}_p = \boldsymbol{w}^T \tilde{\boldsymbol{r}} + (\boldsymbol{w^d})^T \tilde{\boldsymbol{r}}^{\boldsymbol{d}}$.
- $\boldsymbol{w^d} \geq \boldsymbol{0}$; $\boldsymbol{1}^T \boldsymbol{w} + \boldsymbol{1}^T \boldsymbol{w^d} = 1$ - else, too risky & nonconvex

- Robust max-min:

$$\max_{(\boldsymbol{w},\boldsymbol{w^d}) \in \mathcal{W}} \quad \min_{\substack{\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}, \\ \boldsymbol{r^d} = f(\boldsymbol{r})}} \boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w^d})^T \boldsymbol{r^d}$$

- Equivalent SIP:

$$\begin{aligned} \underset{\boldsymbol{w},\boldsymbol{w^d},\phi}{\text{maximise}} \quad & \phi \\ \text{s.t.} \quad & \boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w^d})^T \boldsymbol{r^d} \geq \phi \quad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r^d} = f(\boldsymbol{r}) \\ & (\boldsymbol{w}, \boldsymbol{w^d}) \in \mathcal{W} \end{aligned}$$

- At optimality ϕ^* worst-case portfolio return, $\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}$.

Incorporating Options in Robust Framework

- Portfolio return $\tilde{r}_p = \boldsymbol{w}^T \tilde{\boldsymbol{r}} + (\boldsymbol{w^d})^T \tilde{\boldsymbol{r}}^{\boldsymbol{d}}$.
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- Equivalent SOCP:

$$\max_{\substack{\boldsymbol{w},\boldsymbol{w^d},\phi,\\ \boldsymbol{y},\boldsymbol{s}}} \quad \phi$$

$$\text{s.t.} \quad \boldsymbol{\mu}^T(\boldsymbol{w} + \mathbf{B}^T\boldsymbol{y} - \boldsymbol{s}) - \delta \left\| \Sigma^{1/2}(\boldsymbol{w} + \mathbf{B}^T\boldsymbol{y} - \boldsymbol{s}) \right\|_2 + \boldsymbol{a}^T\boldsymbol{y} \geq \phi$$

$$(\boldsymbol{w}, \boldsymbol{w^d}) \in \mathcal{W}, \ \boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{w^d}, \ \boldsymbol{s} \geq \boldsymbol{0}$$

- At optimality ϕ^* worst-case portfolio return, $\boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}}$.

Insured Robust Portfolio Optimisation

- Non-inferiority guarantee at optimality :

$$\boldsymbol{w}_*^T \boldsymbol{r} + (\boldsymbol{w}_*^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}} \geq \phi^* \qquad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r})$$

- Extreme events: $\tilde{\boldsymbol{r}} \rightarrow$ outside $\mathcal{U}_{\boldsymbol{r}}$ $\rightarrow$ no guarantees!
- Control deterioration below ϕ for *any* realisation $\tilde{\boldsymbol{r}}$:

$$\boldsymbol{w}^T \boldsymbol{r} + (\boldsymbol{w}^{\boldsymbol{d}})^T \boldsymbol{r}^{\boldsymbol{d}} \geq \theta\phi \qquad \forall \boldsymbol{r} \in \mathcal{B},\ \boldsymbol{r}^{\boldsymbol{d}} = f(\boldsymbol{r}), \quad \theta \in [0, 1].$$

- Insurance guarantee expressed as fraction of ϕ:
  - Non-inferiority guarantee is no hedge against extremes
  - Prevents overly expensive insurance.

Guarantee Tradeoff

- Insured robust portfolio optimisation:

$$\begin{aligned}
\max_{\boldsymbol{w},\boldsymbol{w^d},\phi} \quad & \phi \\
\text{subject to} \quad & \boldsymbol{w}^T\boldsymbol{r} + (\boldsymbol{w^d})^T\boldsymbol{r^d} \geq \phi \qquad \forall \boldsymbol{r} \in \mathcal{U}_{\boldsymbol{r}},\ \boldsymbol{r^d} = f(\boldsymbol{r}) \\
& \boldsymbol{w}^T\boldsymbol{r} + (\boldsymbol{w^d})^T\boldsymbol{r^d} \geq \theta\phi \qquad \forall \boldsymbol{r} \in \mathcal{B},\ \boldsymbol{r^d} = f(\boldsymbol{r}) \\
& (\boldsymbol{w}, \boldsymbol{w^d}) \in \mathcal{W}.
\end{aligned}$$

- Has SOCP reformulation $\rightarrow$ tractable.
- Exposes tradeoff: non-inferiority vs insurance guarantees:
  - $\mathcal{U}_{\boldsymbol{r}}$ increases $\Rightarrow \phi^*$ decreases.
  - ϕ^* decreases $\rightarrow \begin{cases} \text{insurance level } \theta\phi^* \\ \text{associated insurance cost/premium} \end{cases}$

Outline

Outline

FX Portfolios - Triangulation

[Fonseca, Wiesemann, Zymler, Kuhn & R, 2011]

- n currencies: E_i domestic/unit ith foreign
- E_i^0 & E_i: today & future spot rate
- $e_i = E_i / E_i^0$: currency i return - uncertain

- EUR/USD & GBP/USD $\Longleftrightarrow$ cross-rate EUR/GBP
- No-arbitrage: non-convex constraint

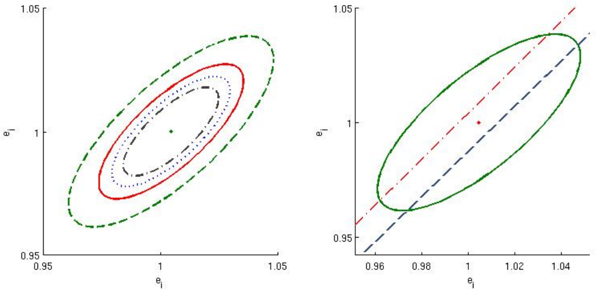
$$\Leftrightarrow e_i \cdot \frac{1}{e_j} \cdot ce_{ij} \quad = \quad 1 \quad \forall i, j = 1, \ldots, n; \quad (ij) = 1, \ldots, \frac{n(n-1)}{2}$$

- Uncertainty interval for cross-rates $ce_{ij} = \frac{e_j}{e_i}$;
- Convex: $n(n-1)$ inequalities for cross rates

$$\begin{aligned} \underline{ce} \leq & \quad ce_{ij} & \leq \bar{ce} \\ \Leftrightarrow \underline{ce} \cdot e_i \leq & \quad e_j & \leq \bar{ce} \cdot e_i, \quad \forall e_i \neq 0 \end{aligned}$$

Currency Return Uncertainty Θ_e

- $\Theta_e = \{\mathbf{e} \geq 0 \mid (\mathbf{e} - \bar{\mathbf{e}})' \Sigma^{-1} (\mathbf{e} - \bar{\mathbf{e}}) \leq \delta^2 \wedge \mathbf{A}\mathbf{e} \geq 0\}$
- A: triangular relationship among rates



Robust Optimisation

$$\max_{w \in \mathbb{R}^n} \min_{e \in \Theta_e} \left\{ \mathbf{w}^T \mathbf{e} \;\middle|\; \mathbf{w}^T \mathbf{1} = 1,\ \mathbf{w},\ \geq 0 \right\}$$

Robust Optimisation

Solution

- To solve problem

$$\max_{w \in \mathbb{R}^n} \min_{\mathbf{e} \in \Theta_e} \mathbf{w}^T \mathbf{e}$$

$$\text{s.t.} \quad \mathbf{w}^T \mathbf{1} = 1$$

$$\mathbf{w} \geq 0$$

Robust Optimisation
Solution

- To solve problem
- Start inner min wrt FX return

$$\max_{w \in \mathbb{R}^n} \quad \min_{\mathbf{e} \in \Theta_e} \quad \mathbf{w}^T \mathbf{e}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{w}^T \mathbf{1} &= 1 \\ \mathbf{w} &\geq 0 \end{aligned}$$

$$\min_{e \in \mathbb{R}^n} \quad \mathbf{w}^T \mathbf{e}$$

$$\begin{aligned} \text{s.t.} \quad \|\Sigma^{-1/2}(\mathbf{e} - \bar{\mathbf{e}})\| &\leq \delta \\ \mathbf{A}\mathbf{e} &\geq 0 \\ \mathbf{e} &\geq 0 \end{aligned}$$

Dual Problem

- SOCP: primal & dual have same objective value.
- Dual:

$$\max_{v,k,y} \quad \bar{\mathbf{e}}^T(\mathbf{w} - \mathbf{s}) - \delta v$$

$$\begin{aligned} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w} - \mathbf{s})\| &= v \\ \mathbf{s} &\leq \mathbf{w} \\ \mathbf{s}, v &\geq 0 \\ \mathbf{A}^T\mathbf{k} + \mathbf{y} &= \mathbf{s} \end{aligned}$$

Dual Problem

- SOCP: primal & dual have same objective value.

- Dual:

$$
\begin{aligned}
\max_{v,k,y} \quad & \bar{\mathbf{e}}^T(\mathbf{w}-\mathbf{s}) - \delta v \\
\text{s. t.} \quad & \|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| = v \\
& \mathbf{s} \le \mathbf{w} \\
& \mathbf{s}, v \ge 0 \\
& \mathbf{A}^T\mathbf{k} + \mathbf{y} = \mathbf{s}
\end{aligned}
$$

- Replace original problem:

$$
\begin{aligned}
\max_{w,k,y} \quad & \phi \\
\text{s. t.} \quad & \bar{\mathbf{e}}^T(\mathbf{w}-\mathbf{s}) - \delta\|\Sigma^{1/2}(\mathbf{w}-\mathbf{s})\| \ge \phi \\
& \mathbf{s} \le \mathbf{w} \\
& \mathbf{w}^T\mathbf{1} = 1 \\
& \mathbf{w}, \mathbf{s} \ge 0 \\
& \mathbf{A}^T\mathbf{k} + \mathbf{y} = \mathbf{s}
\end{aligned}
$$

Robust Hedging

Integrating Options

- Option returns:

$$e^d \equiv f(e) = \max\{0, a_p + b_p e\} \quad , \quad a_p = \frac{K}{p} \quad , \quad b_p = -\frac{E^0}{p}$$

$$\Rightarrow \boxed{e^d \equiv f(e) = \max\left\{0, \frac{K - E^0 e}{p}\right\}}$$

- To guard FX returns outside Θ_e, investing in currency O's, with minimum return guarantee: ρ.

Outline

Robust International Portfolio Optimisation

- n assets & m currencies - both returns uncertain
- Allocation matrix $\mathcal{O}$:

$$o_{ij} = \begin{cases} 1, & \text{if } i\text{th asset in } j\text{th currency} \\ 0, & \text{otherwise} \end{cases}$$

- 2 returns for each asset i:
  - Local asset: $r_a^i = P_i / P_i^0$
  - Currency: $r_e^j = E_j / E_j^0$
- Hedging: Quanto options - linking foreign equity with forward FX

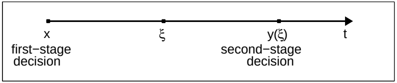
Basic Robust Optimisation

$$\max_{\boldsymbol{w}} \min_{\boldsymbol{r_a}, \boldsymbol{r_e} \in \Xi} \left\{ [\mathrm{diag}(\boldsymbol{r_a}) \mathcal{O} \boldsymbol{r_e}]^T \mathbf{w}) \;\middle|\; \boldsymbol{w}^T \mathbf{1} = 1, \; \boldsymbol{w} \geq 0 \right\}$$

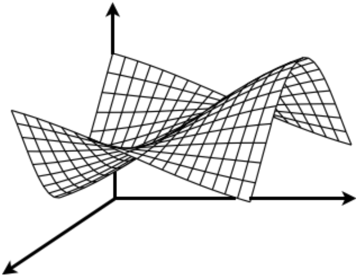
$$\Xi = \left\{ \boldsymbol{r_a}, \boldsymbol{r_e} \geq 0 : \boldsymbol{A r_e} \geq 0 \wedge \left(\begin{bmatrix} \boldsymbol{r_a} \\ \boldsymbol{r_e} \end{bmatrix} - \begin{bmatrix} \bar{\boldsymbol{r}}_{\boldsymbol{a}} \\ \bar{\boldsymbol{r}}_{\boldsymbol{e}} \end{bmatrix} \right)^T \Sigma^{-1} \left(\begin{bmatrix} \boldsymbol{r_a} \\ \boldsymbol{r_e} \end{bmatrix} - \begin{bmatrix} \bar{\boldsymbol{r}}_{\boldsymbol{a}} \\ \bar{\boldsymbol{r}}_{\boldsymbol{e}} \end{bmatrix} \right) \leq \delta^2 \right\}$$

Outline

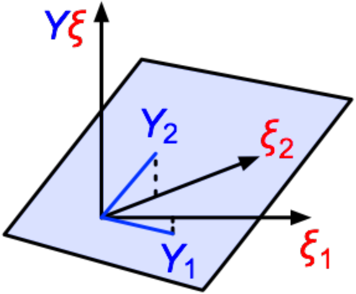
One-Stage to Two-Stages



Approximating nonlinear decision rule by affine rules.



(a) Nonlinear decision rule
$a^T x + c^T y(\xi) \leq b(\xi), \forall \xi \in \Xi$



(b) Affine decision rule
$a^T x + c^T (y_0 + Y\xi) \leq b(\xi),$
$\forall \xi \in \Xi; \quad (y(\xi) \approx y_0 + Y\xi)$

Robust optimisation to reformulate the constraints.

Outline

Outline

Example: Minimax Hedging Strategy

Problem

- Option = contract entitling holder to buy/sell specific # shares at a certain time - for agreed price
- Hedging option risk mainly confined to option seller due to liability contingent on asset underlying option
- Seller of option needs to position to minimise potential negative impact of such liability
- Selling option risky with potentially unlimited loss - buying option mainly nonrisky - insurance at a price and minor risk is its potential loss if option not exercised
- Strategy minimises (by choosing # shares to hold - instead of all contracted shares) worst-case potential hedging error (wrt future stock price).

Example: Minimax Hedging Strategy

Problem formulation[12]:

$$\begin{array}{ll} \min_{x_t} & \max_{y_{t+1}^S} f(x_t, y_{t+1}^S), \\ s.t. & y_{t+1}^{S,lower} \leq y_{t+1}^S \leq y_{t+1}^{S,upper} \end{array}$$

[12] R and Howe [2002]

Example: Hedging Error

Hedging error	:	$HE = N(B_t - B_{t+1}(y^S_{t+1})) + x_t(y^S_{t+1} - y^S_t)$
N	:	contracted # shares
B_t	:	call price
x_t	:	# shares to hold
$y^S_{t+1} \in \mathcal{R}^k$	:	stock price
$U^d \in \mathcal{R}^{k+1}$	:	desired potential HE & transaction c

Minimax hedging strategy

Minimise max potential HE between t & $t+1$:

$$f(x_t, y^S_{t+1}) = \frac{1}{2} < U(x_t, y^S_{t+1}) - U^d, Q(U(x_t, y^S_{t+1}) - U^d) >$$

Example: Hedging Error

- r: risk–free interest rate,
- Δt: hedging interval
- $\hat{K}$: transaction cost (% of transaction volume)

$$
\begin{aligned}
U(x_t, y^S_{t+1}) &= \begin{bmatrix} U_1(x_t, y^S_{t+1}) \\ \vdots \\ U_2(x_t) \end{bmatrix} \\
U_1(x_t, y^S_{t+1}) &= \sum_{i=1}^{k} x_{i,t}(y^S_{i,t+1} - y^S_{i,t}) \\
&+ N_i(B^S_{i,t} - B_{i,t}(y^S_{i,t+1})) \\
&+ \sum_{i=1}^{k}[-(x_{i,t} - x_{i,t-1})y^S_{i,t} \\
&+ C_{i,t-1}(1 + r\Delta t)].r\Delta t
\end{aligned}
$$

Example: Hedging Error

$$\begin{aligned} C_{i,t-1} &= C_{i,t-2}(1 + r\Delta t) \\ &- (x_{i,t-1} - x_{i,t-2})y^S_{i,t-1} \\ &- \hat{K}|(x_{i,t-1} - x_{i,t-2})y^S_{i,t-1}|. \end{aligned}$$

$$U_2(x_t) = \begin{bmatrix} U_{1,2}(x_{1,t}) \\ \vdots \\ U_{k,2}(x_{k,t}) \end{bmatrix}$$

with $U_{i,2}(x_{i,t}) = \hat{K}(x_{i,t} - x_{i,t-1})y^S_{i,t}$.

Outline

Example: Global Strucural Model Distinguishability

Modelling

- Modelling process systems e.g chemical reactors, crystallisation units, fermentation
- Possible to propose more than one mathematical model to describe underlying system
- We wish to determine whether the mathematical structure of these models can be distinguished from one another
- Goal: determine best model
- Two models fermentation of baker's yeast in a batch reactor
- Approximate solution of these models by their 'Fleiss' functional expansions

Example: Global Strucural Model Distinguishability

Structural distinguishability problem[13]:

$$\min_{\theta,\theta^*} \max_{x} \quad \Phi_D = \sum_{k=1}^{4} [L_k^{[1]}(x,\theta) - L_k^{[2]}(x,\theta^*)]^2$$

$$s.t. \quad 10^{-3} \leq \theta_i \leq 1.0, \quad i = 1, ..., 4$$

$$10^{-3} \leq \theta_i^* \leq 1.0, \quad i = 1, ..., 3$$

$$1.0 \leq x_1 \leq 25.0,$$

$$10^{-2} \leq x_2 \leq 25.0$$

- θ & θ^*: parameter vectors from two models
- x: vector of state/response variables.

[13] Žaković and R [2003]

Example: Global Strucural Model Distinguishability

$L_i^k, i = 1,2,3,4, \quad k = 1,2$: coefficients of functional expansions for solution trajectories of two models:
$k = 1$

$$
\begin{aligned}
L_1^{[1]} &= (\frac{\theta_1 x_2}{\theta_2 + x_2} - \theta_4)x_1 \\
L_2^{[1]} &= -(\frac{\theta_1 x_2}{\theta_2 + x_2} - \theta_4)x_1 - x_1 x_2(\frac{\theta_1}{\theta_2 + x_2} - \frac{\theta_1 x_2}{(\theta_2 + x_2)^2}) \\
L_3^{[1]} &= -\frac{\theta_1 x_1 x_2}{\theta_3(\theta_2 + x_2)} \\
L_4^{[1]} &= \frac{\theta_1 x_1 x_2}{\theta_3(\theta_2 + x_2)} - x_1 x_2(-\frac{\theta_1}{\theta_3(\theta_2 + x_2)} + \frac{\theta_1 x_2}{\theta_3(\theta_2 + x_2)^2})
\end{aligned}
$$

Example: Global Strucural Model Distinguishability

$k = 2$

$$
\begin{aligned}
L_1^{[2]} &= (\theta_1^* x_2 - \theta_3^*) x_1 \\
L_2^{[2]} &= -(\theta_1^* x_2 - \theta_3^*) x_1 - x_2 \theta_1^* x_1 \\
L_3^{[2]} &= -\frac{\theta_1^* x_2 x_1}{\theta_2^*} \\
L_4^{[2]} &= 2\frac{\theta_1^* x_2 x_1}{\theta_2^*}
\end{aligned}
$$

Outline

Robust Optimisation: Conclusions

Optimisation Under Uncertainty

- Intuitive approach to data uncertainty
- Immunises against effects of uncertainty
- Out-of-sample improvements with RO
- Non-inferiority property & further guarantees.
- No substitute to wisdom!

Algorithms

- Good algorithms for convex-concave problems.
- Multiple & global optima issues for nonconvex problems.

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