Complex Systems- Exercises 1 (solutions)

1. Find the dominant term and the smallest big O complexity of the following expressions as $x \to \infty$:

- $98x \log x - 23x^{1.1}$.
  Solution: $-23x^{1.1}$ is dominant. $O(x^{1.1})$
- $7x^2 - \frac{4x^3}{\log x}$.
  Solution: $-\frac{4x^3}{\log x}$ is dominant. $O(\frac{x^3}{\log x})$.
- $3 \log_4 x + 2 \log \log x$.
  Solution: $3 \log_4 x$ is dominant. $O(\log x)$.
- $-x^{-0.2} + (x+2)^{\sin x}$.
  Solution: Neither term dominates the other. E.g., for $x = (2n + \frac{1}{2})\pi$ the second term $(x+2)^{\sin x}$ dominates while for $x = (2n + \frac{3}{2})\pi$ the first term $-x^{-0.2}$ dominates. $O(x)$.

2. Decide whether each statement below is true or false as $x \to \infty$ and prove your assertion:

- $-5x^2 + 3x + 2 = O(x^2)$.

  Solution: True. Let's check this one directly using the definition. We have $|-5x^2 + 3x + 2| \leq |-5x^2| + |3x| + 2 = 5|x^2| + 3|x| + 2$. So for $x \geq 1$ we obtain $|-5x^2 + 3x + 2| \leq 10|x^2|$.

- $e^x/100 = O(2^x)$.
  Solution: False. For any $M > 0$ and $x_0 \in \mathbb{R}$, we can find $x > x_0$ such that $e^x/100 > M2^x$, e.g. for any $x > \max\{x_0, (\log 100M)/(-1+\log e)\}$, where log is in base 2.

- $(x^5 + 12x^4 - 3x + 2)/(3 + x^5) = o(1)$.
  Solution: False. Since
  $$\lim_{x\to\infty} (x^5 + 12x^4 - 3x + 2)/(3 + x^5) = 1 \neq 0.$$

- $3x^2 - 4x + 5 \sim x^2$.
  Solution: False. Since $(3x^2 - 4x + 5)/x^2 = 3 - 4/x + 5/x^2 \to 3 \neq 1$ as $x \to \infty$.

- $x^{-1.2} - 5x^{-2} \sim x^{-1.2}$.
  Solution: True. Since $(x^{-1.2} - 5x^{-2})/x^{-1.2} = 1 - 5/x^{0.8} \to 1$ as $x \to \infty$.

- $f_1 = O(g_1)$ and $f_2 = O(g_2) \;\Rightarrow\; f_1 + f_2 = O(g_1 + g_2)$.
  Solution: False. For example put $f_1(x) = f_2(x) = x$ and $g_1(x) = x = -g_2(x)$.

- $f_1 = O(g_1)$ and $f_2 = O(g_2) \;\Rightarrow\; f_1 f_2 = O(g_1 g_2)$.
  Solution: True. Suppose $M_i > 0$ and $x_i \in \mathbb{R}$ satisfy $|f_i(x)| \leq M_i|g_i(x)|$ for $x > x_i$ where $i = 1, 2$. Then for $M = M_1 M_2$ and $x_0 = \max\{x_1, x_2\}$, we have $|f_1(x)f_2(x)| = |f_1(x)||f_2(x)| \leq M_1|g_1(x)| \times M_2|g_2(x)| = M|g_1(x)g_2(x)|$ for $x > x_0$.

- $f = O(g)$ and $g = O(h) \;\Rightarrow\; f = O(h)$.
  Solution: True. The proof is similar to the previous case. (Work it out!)

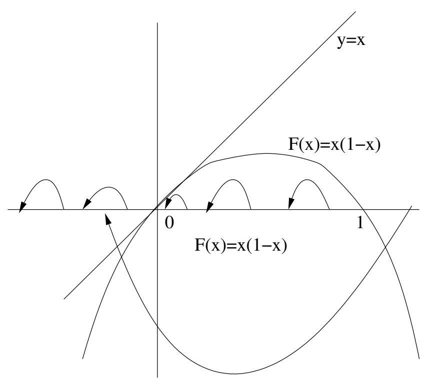
- $f_1 = o(g)$ and $f_2 = o(g) \;\Rightarrow\; f_1 + f + 2 = o(g)$ and $f_1 f_2 = o(g)$.
  Solution: The first statement is true since limits add up:

  $\lim(f_1(x) + f_2(x))/g(x) = \lim f_1(x)/g(x) + \lim f_2(x)/g(x) = 0 + 0 = 0.$

  But the second is false, eg, $f_1(x) = f_2(x) = x$ and $g(x) = x^2$.

3. Determine the type of the fixed points of the map $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x(1-x)$ and sketch its phase portraits.

Solution: We have $F'(x) = 1-2x$. For fixed points we solve $F(x) = x$ i.e., $x - x^2 = x$. There is a unique fixed point $x = 0$ with $F'(0) = 1$, i.e., 0 is non-hyperbolic. It is attracting locally on the right, in fact $[0, 1]$ is its basin of attraction, and repelling on the left.



4. Find all fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3 - \frac{7}{9}x$ and determine whether they are attracting, repelling or neither. Sketch the phase portrait of the map.

Solution: Note $F'(x) = 3x^2 - 7/9$. And $F(x) = x$ gives three fixed points: $x = 0$ is an attracting fixed point ($|F'(0)| = 7/9 < 1$). There are two repelling fixed points at $\pm 4/3$ (($|F'(4/3)| = |F'(-4/3)| = 41/9 > 1$). Orbits of points in $(-4/3, 4/3)$ converge to 0, those in $(4/3, \infty)$ tend to ∞ and those in $(-\infty, -4/3)$ tend to $-\infty$.