Craig Interpolation in Displayable Logics

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Abstract. We give a general proof-theoretic method for establishing Craig interpolation for displayable logics, based upon an analysis of the individual proof rules of their display calculi. Using this uniform method, we establish interpolation for a spectrum of display calculi differing in their structural rules, including those for multiplicative linear logic, multiplicative additive linear logic and ordinary classical logic.
Our analysis at the level of proof rules also provides new insights into the reasons why interpolation fails, or seems likely to fail, in many substructural logics. Specifically, we identify contraction as being particularly problematic for interpolation except in special circumstances.

1 Introduction

> I believe or hope that Display logic can be used as a basis for establishing an interpolation theorem; but that remains to be seen.
>
> Nuel D. Belnap, *Display Logic* [1], 1982

Craig's original interpolation theorem for first-order logic [5] states that for any provable entailment $F \vdash G$ between formulas, an "intermediate formula" or *interpolant* I can be found such that both $F \vdash I$ and $I \vdash G$ are provable and every nonlogical symbol occurring in I occurs in both F and G. This seemingly innocuous property turns out to have considerable mathematical significance because Craig interpolation is intimately connected with consistency, compactness and definability (see [7] for a survey). In computer science, it plays an important rôle in settings in which modular decomposition of complex theories is a concern. In recent years, interpolation has been applied to such various problems as invariant generation [15], type inference [11], model checking [4, 14] and the decomposition of complex ontologies [12]. Thus the question of whether a given logic satisfies *interpolation* is of practical importance in computer science as well as theoretical importance in logic.

In this paper we give a general proof-theoretic method, based on Belnap's *display logic*, for establishing Craig interpolation in propositional logics. Display logic is a general consecution framework which allows us to combine multiple families of logical connectives into a single *display calculus* [1]. Display calculi are characterised by the availability of a "display-equivalence" relation on consecutions which allows us to rearrange a consecution so that a selected substructure appears alone on one side of the proof turnstile. Various authors have shown how to capture large classes of modal and substructural logics within this framework [2, 10, 13, 19], and much work has been done to characterise the class of Kripke frame conditions which can be captured by displayed logics [9]. A major advantage of display calculi is that they enjoy an extremely general cut-elimination theorem which relies on checking eight simple conditions on the rules of the calculus. Restall has also shown how decidability results can be obtained from cut-free display calculi [16].

In the case that a cut-free sequent calculus *à la* Gentzen is available, interpolation can typically be established by induction over cut-free derivations (see e.g. [3]). Besides its theoretical elegance, this method has the advantage that bounds on the size of the interpolant may easily be obtained as a function of the size of the derivation. One of the main criticisms levelled against display calculi is that they do not enjoy a true sub-formula property and hence, in contrast to the situation in sequent calculus, Belnap's general cut-elimination theorem cannot be used to prove results like interpolation for display calculi. Indeed, to our knowledge there are no interpolation theorems in the literature based on display logic. Here we (partially) rebut the aforementioned criticism by giving a general Craig interpolation result for a large class of displayed logics.

Our methodology revolves around the construction of a set of interpolants at each step of the proof, one for every possible "rearrangement" of the consecution under consideration, where the notion of rearrangement is given by the combination of display-equivalence and any native associativity principles for the binary structural connectives. That is, we show that, given interpolants for all rearrangements of the premises, one can find interpolants for all rearrangements of the conclusion. This gives us a very general interpolation method that applies to a wide range of logics with a display calculus presentation, and that is potentially extensible to even bigger classes of logics. However, some proof rules enjoy the aforementioned local property only under strong restrictions, with contraction being the most problematic among the rules we study in this paper. This gives us a significant new insight into the reasons why interpolation fails, or is likely to fail, in many substructural logics.

The remainder of this paper is structured as follows. In Section 2 we introduce the display calculi that we shall work with throughout the paper. Section 3 introduces our general methodology and shows how to apply it to the more straightforward display calculus proof rules. Sections 4 and 5 then respectively treat binary logical rules and structural rules, which are considerably more complicated. Section 6 concludes.

2 Display calculus fundamentals

In this section we give a basic display calculus which can be customised to various logics by adding structural rules. We note that, in general, one may formulate display calculi for logics involving arbitrarily many families of formula and structure connectives. In order to limit the bureaucracy and general technical overhead associated with such generality, however, our display calculi in this paper are limited to those employing only a single family of connectives.

Definition 2.1 (Formula). *Formulas* are given by the following grammar, where P ranges over a fixed infinite set of propositional variables:

$$F ::= P \mid \top \mid \bot \mid \neg F \mid F \,\&\, F \mid F \vee F \mid F \to F \mid \top_a \mid \bot_a \mid F \,\&_a\, F \mid F \vee_a F$$

We write $\mathcal{V}(F)$ to denote the set of propositional variables occurring in F. We write F, G, I etc. to range over formulas. The subscript "a" is for "additive".

Definition 2.2 (Structure / consecution). *Structures* are given by the following grammar, where F ranges over formulas:

$$X ::= F \mid \emptyset \mid \sharp X \mid X \,;\, X$$

A structure is called *atomic* if it is either a formula or $\emptyset$. When we reason by structural induction on a structure X, we typically conflate the cases $X = F$ and $X = \emptyset$ into the case where X is atomic. We write W, X, Y, Z etc. to range over structures, and A, B etc. to range over atomic structures.

If X and Y are structures then $X \vdash Y$ is a *consecution*. We write $\mathcal{C}, \mathcal{C}'$ etc. to range over consecutions.

Definition 2.3 (Antecedent and consequent parts). A *part* of a structure X is an occurrence of one of its substructures. We classify the parts of X as either *positive* or *negative* in X as follows:

- X is a positive part of itself;
- any negative (positive) part of X is a positive (negative) part of $\sharp X$;
- any positive (negative) part of X_1 or of X_2 is a positive (negative) part of the structure $X_1 \,;\, X_2$.

Z is said to be an *antecedent (consequent) part* of a consecution $X \vdash Y$ if it is a positive (negative) part of X or a negative (positive) part of Y.

The following definition gives the proper reading of consecutions as formulas.

Definition 2.4 (Validity). For any structure X we define the formulas Ψ_X and Υ_X by mutual structural induction on X as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\Psi_F &=& F & \Upsilon_F &=& F \\
\Psi_\emptyset &=& \top & \Upsilon_\emptyset &=& \bot \\
\Psi_{\sharp X} &=& \neg \Upsilon_Y & \Upsilon_{\sharp X} &=& \neg \Psi_X \\
\Psi_{X_1;X_2} &=& \Psi_{X_1} \,\&\, \Psi_{X_2} & \Upsilon_{X_1;X_2} &=& \Upsilon_{X_1} \vee \Upsilon_{X_2}
\end{array}$$

$X \vdash Y$ is said to be *valid* in a logic $\mathcal{L}$ iff $\Psi_X \vdash \Upsilon_Y$ is a valid entailment of $\mathcal{L}$.

Identity rules:

$$\frac{}{P \vdash P}\,(\mathrm{Id}) \qquad\qquad \frac{X' \vdash Y'}{X \vdash Y}\; X \vdash Y \equiv_D X' \vdash Y' \;\; (\equiv_D)$$

Logical rules:

$$\frac{\emptyset \vdash X}{\top \vdash X}\,(\top\mathrm{L}) \qquad \frac{}{\emptyset \vdash \top}\,(\top\mathrm{R}) \qquad \frac{F;G \vdash X}{F \,\&\, G \vdash X}\,(\&\mathrm{L}) \qquad \frac{X \vdash F \quad Y \vdash G}{X \,;\, Y \vdash F \,\&\, G}\,(\&\mathrm{R})$$

$$\frac{}{\bot \vdash \emptyset}\,(\bot\mathrm{L}) \qquad \frac{X \vdash \emptyset}{X \vdash \bot}\,(\bot\mathrm{R}) \qquad \frac{F \vdash X \quad G \vdash Y}{F \vee G \vdash X \,;\, Y}\,(\vee\mathrm{L}) \qquad \frac{X \vdash F;G}{X \vdash F \vee G}\,(\vee\mathrm{R})$$

$$\frac{\sharp F \vdash X}{\neg F \vdash X}\,(\neg\mathrm{L}) \qquad \frac{X \vdash \sharp F}{X \vdash \neg F}\,(\neg\mathrm{R}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \rightarrow G \vdash \sharp X \,;\, Y}\,(\rightarrow\mathrm{L}) \qquad \frac{X \,;\, F \vdash G}{X \vdash F \rightarrow G}\,(\rightarrow\mathrm{R})$$

Fig. 1. Proof rules for the basic display calculus $\mathcal{D}_0$.

Definition 2.5 (Display-equivalence). We define *display-equivalence* $\equiv_D$ to be the least equivalence on consecutions containing the (symmetric) relation $\rightleftarrows_D$ given by the following *display postulates*:

$$\begin{array}{ccccc} X;Y \vdash Z & \rightleftarrows_D & X \vdash \sharp Y;Z & \rightleftarrows_D & Y;X \vdash Z \\ X \vdash Y;Z & \rightleftarrows_D & X;\sharp Y \vdash Z & \rightleftarrows_D & X \vdash Z;Y \\ X \vdash Y & \rightleftarrows_D & \sharp Y \vdash \sharp X & \rightleftarrows_D & \sharp\sharp X \vdash Y \end{array}$$

We remark that our notion of display-equivalence builds in the commutativity of ; on the left and right of consecutions, i.e., we are assuming both & and $\vee$ commutative. This makes life slightly easier, but is not crucial to our developments.

Proposition 2.6 (Display property). *For any antecedent (consequent) part Z of a consecution $X \vdash Y$, one can construct a structure W such that $X \vdash Y \equiv_D Z \vdash W$ ($X \vdash Y \equiv_D W \vdash Z$ respectively).*

Proof. (Sketch) The required property follows from the fact that, for any consecution $X \vdash Y$, the display postulates of Defn. 2.5 facilitate the display of each of the *immediate* substructures of X and Y (as the antecedent or consequent as appropriate); it follows that any arbitrary substructure can be displayed by iteration. □

The process of rearranging a consecution $X \vdash Y$ into the consecution $Z \vdash W$ or $W \vdash Z$ via display-equivalence in Prop. 2.6 is called *displaying* Z, and Z is said to be *displayed* in the resulting consecution.

In Figure 1 we give the proof rules of a basic display calculus $\mathcal{D}_0$ which only uses the logical connectives $\top$, $\bot$, $\neg$, &, $\vee$, and $\rightarrow$. Figure 2 presents "structure-free" proof rules for the *additive* logical connectives $\top_a$, $\bot_a$, $\&_a$ and $\vee_a$, and

$$\frac{}{\bot_a \vdash X}\,(\bot_a\mathrm{L}) \qquad \frac{F_i \vdash X}{F_1 \mathbin{\&_a} F_2 \vdash X}\; i \in \{1,2\}\,(\&_a\mathrm{L}) \qquad \frac{F \vdash X \quad G \vdash X}{F \vee_a G \vdash X}\,(\vee_a\mathrm{L})$$

$$\frac{}{X \vdash \top_a}\,(\top_a\mathrm{R}) \qquad \frac{X \vdash F \quad X \vdash G}{X \vdash F \mathbin{\&_a} G}\,(\&_a\mathrm{R}) \qquad \frac{X \vdash F_i}{X \vdash F_1 \vee_a F_2}\; i \in \{1,2\}\,(\vee_a\mathrm{R})$$

Fig. 2. Structure-free proof rules for the "additive" logical connectives.

$$\frac{\emptyset; X \vdash Y}{X \vdash Y}\,(\emptyset\mathrm{C_L}) \qquad \frac{X \vdash Y}{\emptyset; X \vdash Y}\,(\emptyset\mathrm{W_L}) \qquad \frac{X \vdash Y; \emptyset}{X \vdash Y}\,(\emptyset\mathrm{C_R}) \qquad \frac{X \vdash Y}{X \vdash Y; \emptyset}\,(\emptyset\mathrm{W_R})$$

$$\frac{(W;X);Y \vdash Z}{W;(X;Y) \vdash Z}\,(\mathrm{A}) \qquad \frac{X \vdash Z}{X;Y \vdash Z}\,(\mathrm{W}) \qquad \frac{X;X \vdash Y}{X \vdash Y}\,(\mathrm{C})$$

Fig. 3. Some structural rules.

Figure 3 presents some *structural rules* governing the behaviour of the structural connectives $\emptyset$, ';' and $\sharp$. All of these rules should be regarded as optional; if $\mathcal{D}$ is a display calculus and $\mathcal{R}$ is a list of rules drawn from those in Figures 2 and 3 then the *extension* $\mathcal{D}+\mathcal{R}$ of $\mathcal{D}$ is the display calculus obtained from $\mathcal{D}$ by adding all of the rules in $\mathcal{R}$. We write $\mathcal{D}_0^+$ to abbreviate the extension of $\mathcal{D}_0$ with all of the structure-free rules in Figure 2.

Since we will establish interpolation by induction over cut-free derivations, we have omitted the usual cut rule from $\mathcal{D}_0$. The following theorem establishes that this omission is harmless.

Theorem 2.7. *The following cut rule is admissible in any extension of $\mathcal{D}_0$:*

$$\frac{X \vdash F \quad F \vdash Y}{X \vdash Y}\,\textit{(Cut)}$$

Proof. (Sketch) As usual, given the display property (Prop. 2.6), one just verifies that the proof rules in Figures 1, 2 and 3 meet Belnap's conditions C1–C8 guaranteeing cut-elimination [1]. □

Comment 2.8. *Under the translation from consecutions to formulas given by Defn. 2.4, certain of our display calculi can be understood as follows:*

$\mathcal{D}_{\mathrm{MLL}} = \mathcal{D}_0 + (\mathrm{A}), (\emptyset\mathrm{C_L}), (\emptyset\mathrm{C_R}), (\emptyset\mathrm{W_L}), (\emptyset\mathrm{W_R})$ *is multiplicative linear logic.*
$\mathcal{D}_{\mathrm{MALL}} = \mathcal{D}_0^+ + (\mathrm{A}), (\emptyset\mathrm{C_L}), (\emptyset\mathrm{C_R}), (\emptyset\mathrm{W_L}), (\emptyset\mathrm{W_R})$ *is multiplicative additive linear logic;*
$\mathcal{D}_{\mathrm{CL}} = \mathcal{D}_0 + (\mathrm{A}), (\emptyset\mathrm{C_L}), (\emptyset\mathrm{C_R}), (\mathrm{W}), (\mathrm{C})$ *is standard classical propositional logic.*

3 Interpolation: unary and structure-free rules

We now turn to the main topic of this paper, the question of whether *interpolation* holds in our display calculi.

Definition 3.1 (Interpolation). A display calculus $\mathcal{D}$ is said to have the *interpolation* property if for any $\mathcal{D}$-provable consecution $X \vdash Y$ one can find an *interpolant*, defined as a formula I such that $X \vdash I$ and $I \vdash Y$ are both $\mathcal{D}$-provable and such that $\mathcal{V}(I) \subseteq \mathcal{V}(X) \cap \mathcal{V}(Y)$, where $\mathcal{V}(Z)$ denotes the set of propositional variables occurring in the structure Z.

We note that, by cut-admissibility (Theorem 2.7), the existence of an interpolant for a consecution $\mathcal{C}$ implies the provability of $\mathcal{C}$.

We aim to emulate the spirit of the classical proof-theoretic approach to interpolation for cut-free sequent calculi such as Gentzen's LK (see e.g. [3]). That is, given a cut-free display calculus proof of a consecution, we aim to construct its interpolant by induction over the structure of the proof. However, the display postulates introduce a difficulty: for example, given an interpolant for $X ; Y \vdash Z$, it is not clear how to use it to obtain an interpolant for $X \vdash \sharp Y ; Z$. In fact, similar problems arise for sequent calculi as well (e.g., in the classical negation rules of LK), and the usual solution is to simultaneously construct interpolants for *all possible decompositions* of each sequent. We shall employ an analogue of this strategy for the setting of display calculi: we shall simultaneously construct interpolants for *all possible rearrangements* of each consecution, where the notion of "rearrangement" is provided by the combination of display-equivalence and, if it is present in the calculus, the associativity rule (A). The latter inclusion is necessary for similar reasons to those for the inclusion of the display postulates. (A similar combination of display-equivalence and associativity was employed by Restall in his work on decidable display calculi for relevant logics [16]).

Definition 3.2. Let $\mathcal{D}$ be a display calculus and $\mathcal{C}$, $\mathcal{C}'$ be consecutions. We define $\mathcal{C} \to_A \mathcal{C}'$ to hold iff $\mathcal{D}$ includes (A) and $\mathcal{C}$ is the premise of an instance of (A) with conclusion $\mathcal{C}'$. Then the relation $\to_{AD}$ is defined to be $\to_A \cup \rightleftarrows_D$ and the relation $\equiv_{AD}$ is defined to be the reflexive-transitive closure of $\to_{AD}$.

Clearly $\equiv_D \subseteq \equiv_{AD}$ and $\equiv_{AD}$ is $\equiv_D$ in any display calculus without (A).

Comment 3.3. *The relation $\equiv_{AD}$ is indeed an equivalence relation, because the reverse direction of $\to_A$ is included in $\equiv_{AD}$ via the following:*

$$W;(X;Y) \vdash Z \equiv_D (Y;X);W \vdash Z \to_A Y;(X;W) \vdash Z \equiv_D (W;X);Y \vdash Z$$

Furthermore, the following proof rule ($\equiv_{AD}$) is derivable in any extension of $\mathcal{D}_0$:

$$\frac{X' \vdash Y'}{X \vdash Y} \; X \vdash Y \equiv_{AD} X' \vdash Y' \;\; (\equiv_{AD})$$

Our definition of $\equiv_{AD}$ gives rise to the following "local interpolation" property for display calculus proof rules.

Definition 3.4 (LADI property). A proof rule of a display calculus $\mathcal{D}$ with conclusion $\mathcal{C}$ is said to have the *local AD-interpolation* (LADI) property if for each premise $\mathcal{C}_i$ we have interpolants for all $\mathcal{C}'_i \equiv_{AD} \mathcal{C}_i$, we can construct interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$.

Lemma 3.5. *If the proof rules of a display calculus $\mathcal{D}$ each have the LADI property, then $\mathcal{D}$ has the interpolation property.*

Proof. We must show any $\mathcal{D}$-provable consecution $\mathcal{C}$ has an interpolant. We prove by induction on the proof of $\mathcal{C}$ that we have interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$, using LADI for the proof rules at each induction step. In particular, this yields an interpolant for $\mathcal{C}$. $\square$

Thus the LADI property gives a sufficient condition, in terms of individual proof rules, for interpolation to hold in display calculi. In the remainder of this section, we shall show that this property holds for the single-premise rules of $\mathcal{D}_0$ and the structure-free rules for the additives in Figure 2. Then, in later sections, we shall examine the situation for the two-premise rules of $\mathcal{D}_0$ and for the structural rules in Figure 3.

In our proofs, it will be essential to keep track of the atomic parts of a consecution being shuffled around using $\equiv_{AD}$, and possibly substitute other structures for these parts. It is intuitively obvious how to do this; the next definitions are intended to formalise the concept.

Definition 3.6 (Substitution). Let Z be a part of the structure X. We write the substitution notation $X[Y/Z]$, where Y is a structure, to denote the replacement of Z (which we emphasise is a substructure *occurrence*) by the structure Y. We extend substitution to consecutions in the obvious way.

Definition 3.7 (Congruence). Let $\mathcal{C} \to_{AD} \mathcal{C}'$, whence $\mathcal{C}$ and $\mathcal{C}'$ are obtained by assigning structures to the structure variables occurring in our statement of some display postulate (see Defn. 2.5) or the rule (A) (see Figure 3). Two atomic parts A and A' of $\mathcal{C}$ and $\mathcal{C}'$ respectively are said to be *congruent* if they occupy the same position in the structure assigned to some structure variable. For example, the two indicated occurrences of F are congruent in $X; (F; \emptyset) \vdash Z \to_{AD} X \vdash \sharp(F; \emptyset); Z$, as are the two indicated occurrences of $\emptyset$, because they occupy the same position in the structure $(F; \emptyset)$ assigned to the structure variable Y in our statement of the display postulate $X; Y \vdash Z \rightleftarrows_D X \vdash \sharp Y; Z$.

We extend congruence to atomic parts of consecutions related by $\equiv_{AD}$ by reflexive-transitive induction on $\equiv_{AD}$ as follows:

- If $\mathcal{C} = \mathcal{C}'$ then any part of $\mathcal{C}$ is congruent to itself.
- If $\mathcal{C} \to_{AD} \mathcal{C}'' \equiv_{AD} \mathcal{C}'$ then parts Z and Z' of $\mathcal{C}$ and $\mathcal{C}'$ respectively are congruent if there is a part Z'' of $\mathcal{C}''$ such that Z is congruent to Z'' and Z'' is congruent to Z'.

Finally, we extend congruence to non-atomic parts of consecutions as follows. If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and Z, Z' are parts of $\mathcal{C}, \mathcal{C}'$ respectively then Z and Z' are congruent if every atomic part A of Z is congruent to an atomic part A' of Z', such that the position of A in Z is identical to the position of A' in Z'.

Comment 3.8. *If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ then, for any atomic part A of $\mathcal{C}$, there is a unique congruent atomic part of $\mathcal{C}'$. Moreover, congruent parts of $\mathcal{C}$ and $\mathcal{C}'$ must be occurrences of the same structure.*

For convenience, we usually drop the explicit mention of congruent parts of $\equiv_{AD}$-related consecutions by using identical names for the congruent parts. E.g., when we write $\mathcal{C}[Z/A] \equiv_{AD} \mathcal{C}'[Z/A]$, we mean that the two indicated occurrences of A are congruent[3].

Lemma 3.9 (Substitution lemma). *If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and A is an atomic part of $\mathcal{C}$ then, for any structure Z, we have $\mathcal{C}[Z/A] \equiv_{AD} \mathcal{C}'[Z/A]$.*

Proof. Since the display postulates and the associativity rule (A) are each closed under substitution of an arbitrary structure for congruent atomic parts, this follows by an easy reflexive-transitive induction on $\mathcal{C} \equiv_{AD} \mathcal{C}'$. $\square$

Proposition 3.10. *The proof rules ($\equiv_D$), (Id), ($\top L$), ($\top R$), ($\bot L$), ($\bot R$), ($\neg L$), ($\neg R$), ($\& L$), ($\vee R$), and ($\rightarrow R$) each have the LADI property in any extension of $\mathcal{D}_0$. Furthermore, the associativity rule* (A) *has the LADI property in any extension of $\mathcal{D}_0 +$(A), and the structure-free rules ($\top_a R$), ($\bot_a L$), ($\&_a L$), ($\&_a R$), ($\vee_a L$), and ($\vee_a R$) each have the LADI property in any extension of $\mathcal{D}_0^+$.*

Proof. We treat each proof rule separately, grouping together the rules for which the arguments are similar.

Case ($\equiv_D$).

$$\frac{X' \vdash Y'}{X \vdash Y}\ \ X \vdash Y \equiv_D X' \vdash Y' \ \ (\equiv_D)$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} X' \vdash Y'$, and we require to find interpolants for all $W \vdash Z \equiv_{AD} X \vdash Y$. For any such $W \vdash Z$ we have $W \vdash Z \equiv_{AD} X \vdash Y \equiv_D X' \vdash Y'$ by assumption, and thus $W \vdash Z \equiv_{AD} X' \vdash Y'$ because $\equiv_D \subseteq \equiv_{AD}$ (and $\equiv_{AD}$ is transitive). Thus we are done by the case assumption.

Case (A).

$$\frac{(W;X);Y \vdash Z}{W;(X;Y) \vdash Z}\ \text{(A)}$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} (W;X);Y \vdash Z$ and we require to find interpolants (in $\mathcal{D}_0 +$ (A)) for all $U \vdash V \equiv_{AD} W;(X;Y) \vdash Z$. For any such $U \vdash V$ we have $U \vdash V \equiv_{AD} W;(X;Y) \vdash Z \equiv_{AD} (W;X);Y \vdash Z$ in $\mathcal{D}_0 +$ (A), so are done by the case assumption.

[3] We observe that any ambiguity between different occurrences of the same structure can, in principle, be resolved by uniquely labelling all such occurrences.

Cases (Id), ($\bot$L), ($\top$R). We just show the case of (Id) here; the other two cases are similar.

$$\frac{}{P \vdash P}\,(\mathrm{Id})$$

We require to find interpolants for all $W \vdash Z \equiv_{AD} P \vdash P$, and proceed by reflexive-transitive induction on $\equiv_{AD}$. In the reflexive case we have $W \vdash Z = P \vdash P$ and choose the interpolant to be P, which obviously satisfies the conditions on interpolants. In the transitive case we have $W \vdash Z \to_{AD} W' \vdash Z' \equiv_{AD} P \vdash P$, whence by induction hypothesis we have an interpolant I for $W' \vdash Z'$. We distinguish cases on $W \vdash Z \to_{AD} W' \vdash Z'$. Since $P \vdash P$ contains no semicolons, it is clear by inspection of the display postulates and the associativity rule that neither can $W' \vdash Z'$. Thus $W \vdash Z \to_{AD} W' \vdash Z'$ arises by applying a display postulate from the block:

$$X \vdash Y \rightleftarrows_D \sharp Y \vdash \sharp X \rightleftarrows_D \sharp\sharp X \vdash Y$$

We show a typical case, $X \vdash Y \to_{AD} \sharp Y \vdash \sharp X$. In this case, we have by induction hypothesis that $\mathcal{V}(I) \subseteq \mathcal{V}(\sharp Y) \cap \mathcal{V}(\sharp X)$, and $\sharp Y \vdash I$ and $I \vdash \sharp X$ are both $\mathcal{D}_0$-provable. We choose $\neg I$ to be the interpolant for $X \vdash Y$. Clearly the variable condition is satisfied because $\mathcal{V}(\sharp X) = \mathcal{V}(X)$ and $\mathcal{V}(\sharp Y) = \mathcal{V}(Y)$. For the provability conditions we proceed as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ I \vdash \sharp X\end{array}}{X \vdash \sharp I}\,(\equiv_D)}{X \vdash \neg I}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \sharp Y \vdash I\end{array}}{\sharp I \vdash Y}\,(\equiv_D)}{\neg I \vdash Y}\,(\neg\mathrm{L})$$

The other display postulate cases are similar.

Cases ($\top_a$R), ($\bot_a$L). We show the case of ($\bot_a$L): the case of ($\top_a$R) is similar.

$$\frac{}{\bot_a \vdash X}$$

We require to produce interpolants for all $W \vdash Z \equiv_{AD} \bot_a \vdash X$. Suppose the indicated $\bot_a$ occurs in Z (as a negative part). We pick the interpolant $\neg\bot_a$, which trivially satisfies the variable condition. We have that $W \vdash \neg\bot_a$ and $\neg\bot_a \vdash Z$ are derivable as follows:

$$\dfrac{\dfrac{\dfrac{}{\bot_a \vdash \sharp W}\,(\bot_a\mathrm{L})}{W \vdash \sharp\bot_a}\,(\equiv_D)}{W \vdash \neg\bot_a}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{}{\bot_a \vdash U}\,(\bot_a\mathrm{L})}{\neg\bot_a \vdash Z}\,(\equiv_D)$$

In the right hand derivation, we use the fact that the indicated $\bot_a$ in $\bot_a \vdash X$ is an antecedent part of $\neg\bot_a \vdash Z$ because it is assumed to occur in Z, whence we have $\neg\bot_a \vdash Z \equiv_D \bot_a \vdash U$ for some U by the display property (Prop. 2.6).

If the indicated $\bot_a$ does not occur in Z then it instead occurs in W, in which case we pick interpolant $\bot_a$ and the argument is similar to the above. This completes the case.

Cases ($\top$L), ($\bot$R), ($\neg$L), ($\neg$R), (&L), ($\vee$R), ($\rightarrow$R), ($\&_a$L), ($\vee_a$R). We show the case of (&L); the other cases are similar.

$$\dfrac{F; G \vdash X}{F \,\&\, G \vdash X}\ (\&\mathrm{L})$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} F; G \vdash X$, and we require to find interpolants for all $W \vdash Z \equiv_{AD} F \,\&\, G \vdash X$.

If $W \vdash Z \equiv_{AD} F \,\&\, G \vdash X$ then, by Lemma 3.9, we have that

$$(W \vdash Z)[F; G/F \,\&\, G] \equiv_{AD} F; G \vdash X$$

(where $F\&G$ is the indicated occurrence in $F \,\&\, G \vdash X$). Let I be the interpolant for $(W \vdash Z)[F; G/F \,\&\, G]$ given by assumption. We claim that I is also an interpolant for $W \vdash Z$.

We assume that the indicated $F \,\&\, G$ occurs in W; the case where it instead occurs in Z is similar. In this case we have $(W \vdash Z)[F; G/F \,\&\, G] = W[F; G/F \,\&\, G] \vdash Z$. By assumption we have $W[F; G/F \,\&\, G] \vdash I$ and $I \vdash Z$ provable with $\mathcal{V}(I) \subseteq \mathcal{V}(W[F; G/F \,\&\, G]) \cap \mathcal{V}(Z)$. Then we have $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required since clearly $\mathcal{V}(W) = \mathcal{V}(W[F; G/F \,\&\, G])$. Since $I \vdash Z$ is provable by assumption, it just remains to show that $W \vdash I$ is provable. The required proof is constructed as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{W[F; G/F \,\&\, G] \vdash I}}{F; G \vdash S}\ (\equiv_D)}{\dfrac{F \,\&\, G \vdash S}{W \vdash I}\ (\equiv_D)}\ (\&\mathrm{L})$$

where S is a placeholder for a consequent structure obtained by displaying the indicated $F \,\&\, G$ in $W \vdash I$. The fact that $W[F; G/F \,\&\, G] \vdash I \equiv_D F; G \vdash S$ follows from the fact that $W \vdash I \equiv_D F \,\&\, G \vdash S$ and Lemma 3.9. This completes the case.

Cases ($\&_a$R), ($\vee_a$L). We show the case of ($\vee_a$L); the case of ($\&_a$R) is similar.

$$\dfrac{F \vdash X \quad G \vdash X}{F \vee_a G \vdash X}$$

By assumption we have interpolants for all $W_1 \vdash Z_1 \equiv_{AD} F \vdash X$ and for all $W_2 \vdash Z_2 \equiv_{AD} G \vdash X$, and we must produce interpolants for all $W \vdash Z \equiv_{AD} F \vee_a G \vdash X$.

Suppose that the indicated $F \vee_a G$ occurs in Z. Using the fact that $F \vee_a G \vdash X \equiv_{AD} W \vdash Z$, we have by Lemma 3.9 :

$$\begin{array}{l} F \vdash X \equiv_{AD} (W \vdash Z)[F/F \vee_a G] = W \vdash Z[F/F \vee_a G] \\ G \vdash X \equiv_{AD} (W \vdash Z)[G/F \vee_a G] = W \vdash Z[G/F \vee_a G] \end{array}$$

Let I_1 and I_2 be the interpolants given by assumption for $W \vdash Z[F/F \vee_a G]$ and $W \vdash Z[G/F \vee_a G]$ respectively. We claim that $\neg(\neg I_1 \vee_a I_2)$ is an interpolant[4] for $W \vdash Z$. First we check the variable condition. We have by assumption:

$$\begin{array}{c} \mathcal{V}(I_1) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z[F/F \vee_a G]) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z) \\ \mathcal{V}(I_2) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z[G/F \vee_a G]) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z) \end{array}$$

Thus clearly we have $\mathcal{V}(\neg(\neg I_1 \vee_a \neg I_2)) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required. Next we check the provability conditions. Given that $W \vdash I_1$ and $W \vdash I_2$ are provable by assumption, we can derive $W \vdash \neg(\neg I_1 \vee_a \neg I_2)$ as follows:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$W \vdash I_1$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp I_1 \vdash \sharp W$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$\neg I_1 \vdash \sharp W$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$W \vdash I_2$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp I_2 \vdash \sharp W$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$\neg I_2 \vdash \sharp W$}
\RightLabel{$(\vee_a\mathrm{L})$}
\BinaryInfC{$\neg I_1 \vee_a \neg I_2 \vdash \sharp W$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$W \vdash \sharp(\neg I_1 \vee_a \neg I_2)$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$W \vdash \neg(\neg I_1 \vee_a \neg I_2)$}
\end{prooftree}$$

Finally, given that $I_1 \vdash Z[F/F \vee_a G]$ and $I_2 \vdash Z[G/F \vee_a G]$ are provable by assumption, we must show that $\neg(\neg I_1 \vee_a \neg I_2) \vdash Z$ is provable. First, since the indicated $F \vee_a G$ occurs in Z by assumption, we have $\neg(\neg I_1 \vee_a \neg I_2) \vdash Z \equiv_D F \vee_a G \vdash U$ for some U. Thus by Lemma 3.9 we have:

$$\begin{array}{c} \neg(\neg I_1 \vee_a \neg I_2) \vdash Z[F/F \vee_a G] \equiv_D F \vdash U \\ \neg(\neg I_1 \vee_a \neg I_2) \vdash Z[G/F \vee_a G] \equiv_D G \vdash U \end{array}$$

Using these display-equivalences, we can derive $\neg(\neg I_1 \vee_a \neg I_2) \vdash Z$ as follows:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$I_1 \vdash Z[F/F \vee_a G]$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp Z[F/F \vee_a G] \vdash \sharp I_1$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$\sharp Z[F/F \vee_a G] \vdash \neg I_1$}
\RightLabel{$(\vee_a\mathrm{R})$}
\UnaryInfC{$\sharp Z[F/F \vee_a G] \vdash \neg I_1 \vee_a \neg I_2$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp(\neg I_1 \vee_a \neg I_2) \vdash Z[F/F \vee_a G]$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$\neg(\neg I_1 \vee_a \neg I_2) \vdash Z[F/F \vee_a G]$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$F \vdash U$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$I_2 \vdash Z[G/F \vee_a G]$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp Z[G/F \vee_a G] \vdash \sharp I_2$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$\sharp Z[G/F \vee_a G] \vdash \neg I_2$}
\RightLabel{$(\vee_a\mathrm{R})$}
\UnaryInfC{$\sharp Z[G/F \vee_a G] \vdash \neg I_1 \vee_a \neg I_2$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\sharp(\neg I_1 \vee_a \neg I_2) \vdash Z[G/F \vee_a G]$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$\neg(\neg I_1 \vee_a \neg I_2) \vdash Z[G/F \vee_a G]$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$G \vdash U$}
\RightLabel{$(\vee_a\mathrm{L})$}
\BinaryInfC{$F \vee_a G \vdash U$}
\RightLabel{$(\equiv_D)$}
\UnaryInfC{$\neg(\neg I_1 \vee_a \neg I_2) \vdash Z$}
\end{prooftree}$$

[4] $I_1 \,\&_a\, I_2$ would also work, but this solution allows us to find an interpolant for $(\vee_a\mathrm{L})$ even when $\&_a$ is not a connective of the logic.

This completes the subcase when the indicated $F \vee_a G$ occurs in Z. If it instead occurs in W, then we pick the interpolant to be $I_1 \vee_a I_2$ and the argument is similar. This completes the case, and the proof. $\square$

4 Interpolation: binary logical rules

In this section we extend our basic method for establishing local AD-interpolation of display calculus proof rules to the binary logical rules of $\mathcal{D}_0$. These cases are considerably more complex than the simple 0-ary, unary and structure-free logical rules treated in the previous section, and we will rely heavily on various substitutivity properties of $\equiv_{AD}$ developed here.

The following notion of deletion of a part of a structure or consecution is similar to the one used by Restall in [16]. Note that we write $\sharp^n$ to abbreviate a string of n occurrences of $\sharp$.

Definition 4.1 (Deletion). We say that a part Z of a structure X is *delible from* X if X is not of the form $\sharp^n Z$ for some $n \geq 0$, i.e., X contains a substructure occurrence of the form $\sharp^n Z; W$ (up to commutativity of semicolon). If Z is delible from X then we write $X \setminus Z$ for the structure $X[W/(\sharp^n Z; W)]$, the result of deleting Z from X.

A part Z of a consecution $\mathcal{C}$ is delible from $\mathcal{C}$ if it can be deleted from the side of $\mathcal{C}$ of which it is a part, and we write $\mathcal{C} \setminus Z$ for the consecution obtained by deleting Z from the appropriate side of $\mathcal{C}$.

Lemma 4.2 (Deletion lemma). *Let $\mathcal{C}$ be a consecution and let A be an atomic part of $\mathcal{C}$. If $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and A is delible from $\mathcal{C}$ then the following hold:*

1. *if A is delible from $\mathcal{C}'$ then $\mathcal{C} \setminus A \equiv_{AD} \mathcal{C}' \setminus A$;*
2. *if A is not delible from $\mathcal{C}'$ then one side of $\mathcal{C}'$ is of the form $\sharp^m(Z_1; Z_2)$ and we have $\mathcal{C} \setminus A \equiv_{AD} Z_1 \vdash \sharp Z_2$ if $(Z_1; Z_2)$ is an antecedent part of $\mathcal{C}'$, and $\mathcal{C} \setminus A \equiv_{AD} \sharp Z_1 \vdash Z_2$ if $(Z_1; Z_2)$ is a consequent part of $\mathcal{C}'$.*

Proof. By reflexive-transitive induction on $\mathcal{C} \equiv_{AD} \mathcal{C}'$. In the reflexive case we have $\mathcal{C}' = \mathcal{C}$ and are trivially done. In the transitive case we have $\mathcal{C} \equiv_{AD} \mathcal{C}'' \rightarrow_{AD} \mathcal{C}'$, and we distinguish cases on $\mathcal{C}'' \rightarrow_{AD} \mathcal{C}'$. We show two typical display postulate cases and the associativity case.

Case $S \vdash T \rightarrow_{AD} \sharp S \vdash \sharp T$. For 1, we assume that A is delible from $\sharp S \vdash \sharp T$ whence it is also delible from $S \vdash T$. Thus by induction hypothesis we have $\mathcal{C} \setminus A \equiv_{AD} (S \vdash T) \setminus A$. It is easy to see that $(S \vdash T) \setminus A \equiv_{AD} (\sharp T \vdash \sharp S) \setminus A$ so that $\mathcal{C} \setminus A \equiv_{AD} (\sharp T \vdash \sharp S) \setminus A$ as required.

For 2, suppose A is not delible from $\sharp S \vdash \sharp T$. Then A cannot be delible from $S \vdash T$ either, so by induction hypothesis one side of $S \vdash T$ is of the form $\sharp^m(Z_1; Z_2)$, whence one side of $\sharp T \vdash \sharp S$ is then of the form $\sharp^{m+1}(Z_1; Z_2)$ as required. Assuming that $Z_1; Z_2$ is an antecedent part of $S \vdash T$ (the other case is similar), it is also an antecedent part of $\sharp T \vdash \sharp S$ and thus we have $\mathcal{C} \setminus A \equiv_{AD} Z_1 \vdash \sharp Z_2$ by the induction hypothesis as required.

Case $(S;T);U) \vdash V \rightarrow_{AD} S;(T;U) \vdash V$. For 1, assume that A is delible from $S;(T;U) \vdash V$. Then it is clearly also delible from $(S;T);U \vdash V$, and using the induction hypothesis we easily have as required

$$\mathcal{C} \setminus A \equiv_{AD} ((S;T);U \vdash V) \setminus A \equiv_{AD} (S;(T;U) \vdash V) \setminus A$$

For 2, suppose A is not delible from $S;(T;U) \vdash V$, whence we must have $V = \sharp^n A$ and A is clearly not delible from $(S;T);U \vdash V$ either. Using the induction hypothesis we have one side of $(S;T);U \vdash V$ of the form $\sharp^j(W_1;W_2)$, which forces $j = 0$, $W_1 = (S;T)$ and $W_2 = U$ with $W_1;W_2$ an antecedent part of $(S;T);U \vdash V$. Thus by induction hypothesis we have $\mathcal{C} \setminus A \equiv_{AD} S;T \vdash \sharp U$. Then note that we have one side of $S;(T;U) \vdash V$ of the form $\sharp^m(Z_1;Z_2)$ by taking $m = 0$, $Z_1 = S$ and $Z_2 = (T;U)$. Then we have

$$\mathcal{C} \setminus A \equiv_{AD} S;T \vdash \sharp U \equiv_D S \vdash \sharp(T;U) = Z_1 \vdash \sharp Z_2$$

where $Z_1;Z_2$ is an antecedent part of $S;(T;U) \vdash V$, so that 2 holds as required.

Case $S;T \vdash U \rightarrow_{AD} S \vdash \sharp T;U$. For 1, assume that A is delible from $S \vdash \sharp T;U$. There are two subcases. First, if A is delible from $S;T \vdash U$ then we can easily show using the induction hypothesis that:

$$\mathcal{C} \setminus A \equiv_{AD} (S;T \vdash U) \setminus A \equiv_{AD} (S \vdash \sharp T;U) \setminus A$$

and are done. If A is not delible from $S;T \vdash U$ then we must have $U = \sharp^n A$, whence we have by induction hypothesis that $\mathcal{C} \setminus A \equiv_{AD} S \vdash \sharp T$ (because $S;T$ is an antecedent part of $S;T \vdash U$). Then 1 holds as required because we have

$$\mathcal{C} \setminus A \equiv_{AD} S \vdash \sharp T = (S \vdash \sharp T;\sharp^n A) \setminus A = (S \vdash \sharp T;U) \setminus A$$

For 2, assume that A is not delible from $S \vdash \sharp T;U$, which implies that $S = \sharp^n A$. In that case A is delible from $S;T \vdash U$, so by induction hypothesis we have

$$\mathcal{C} \setminus A \equiv_{AD} (S;T \vdash U) \setminus A = (\sharp^n A;T \vdash U) \setminus A = T \vdash U$$

Thus by taking $m = 0$, $Z_1 = \sharp T$ and $Z_2 = U$ we have that one side of $S \vdash \sharp T;U$ is of the form $\sharp^m(Z_1;Z_2)$ where

$$\mathcal{C} \setminus A \equiv_{AD} T \vdash U \equiv_D \sharp\sharp T \vdash U = \sharp Z_1 \vdash Z_2$$

so that 2 holds as required. This completes the case, and the proof. $\Box$

Lemma 4.3 (Substitutivity I). *For all structures* W, X, Y, Z, *if* $W \vdash X \equiv_{AD} W \vdash Y$ *then* $Z \vdash X \equiv_{AD} Z \vdash Y$, *and if* $X \vdash W \equiv_{AD} Y \vdash W$ *then* $X \vdash Z \equiv_{AD} Y \vdash Z$.

Proof. By Lemma 3.9 it suffices to consider the case in which Z is a formula F. We prove both implications simultaneously by structural induction on W. In each case we just show how to establish the first implication; the second is similar.

Case W atomic. Immediate by Lemma 3.9.

Case $W = \sharp W'$. Using the lemma assumption we have

$$\sharp X \vdash W' \equiv_D \sharp W' \vdash X \equiv_{AD} \sharp W' \vdash Y \equiv_D \sharp Y \vdash W'$$

Thus $\sharp X \vdash W' \equiv_{AD} \sharp Y \vdash W'$ so, using the part of the induction hypothesis given by the second implication, we have

$$\sharp F \vdash X \equiv_D \sharp X \vdash F \equiv_{AD} \sharp Y \vdash F \equiv_D \sharp F \vdash Y$$

Thus $\sharp F \vdash X \equiv_{AD} \sharp F \vdash Y$ so, by Lemma 3.9, we have

$$\sharp\sharp F \vdash X = (\sharp F \vdash X)[\sharp F/F] \equiv_{AD} (\sharp F \vdash Y)[\sharp F/F] = \sharp\sharp F \vdash Y$$

From this we easily have as required:

$$F \vdash X \equiv_{AD} \sharp\sharp F \vdash X \equiv_{AD} \sharp\sharp F \vdash Y \equiv_{AD} F \vdash Y$$

Case $W = W_1; W_2$. Using the lemma assumption we have the following:

$$W_1 \vdash \sharp W_2; X \equiv_D W_1; W_2 \vdash X \equiv_{AD} W_1; W_2 \vdash Y \equiv_D W_1 \vdash \sharp W_2; Y$$

Thus, using (the first part of) the induction hypothesis, we have

$$W_2 \vdash \sharp F; X \equiv_D F \vdash \sharp W_2; X \equiv_{AD} F \vdash \sharp W_2; Y \equiv_D W_2 \vdash \sharp F; Y$$

Applying the induction hypothesis again, we obtain

$$F; G \vdash X \equiv_D G \vdash \sharp F; X \equiv_{AD} G \vdash \sharp F; Y \equiv_D F; G \vdash Y$$

Then, by Lemma 4.2, we have as required

$$F \vdash X = (F; G \vdash X) \setminus G \equiv_{AD} (F; G \vdash Y) \setminus G = F \vdash Y$$

□

Definition 4.4. *Let $\mathcal{C} \equiv_{AD} \mathcal{C}'$ and let Z, Z' be parts of $\mathcal{C}$ and $\mathcal{C}'$ respectively. We say Z' is* built from *Z, written $Z' \lhd Z$, if every atomic part of Z' is congruent to an atomic part of Z.*

Lemma 4.5 (Substitutivity B). *For any structures W, W', X, Y and for any atomic structure A, all of the following hold:*

1. *if $W \vdash X \equiv_{AD} W' \vdash Y$ and $W' \lhd W$ then $\exists U.\ W \vdash A \equiv_{AD} W' \vdash U$;*
2. *if $X \vdash W \equiv_{AD} W' \vdash Y$ and $W' \lhd W$ then $\exists U.\ A \vdash W \equiv_{AD} W' \vdash U$;*
3. *if $W \vdash X \equiv_{AD} Y \vdash W'$ and $W' \lhd W$ then $\exists U.\ W \vdash A \equiv_{AD} U \vdash W'$;*
4. *if $X \vdash W \equiv_{AD} Y \vdash W'$ and $W' \lhd W$ then $\exists U.\ A \vdash W \equiv_{AD} U \vdash W'$.*

(Moreover, in each case we still have $W' \lhd W$ under the replacement of X by A.)

Proof. We show all four implications simultaneously by structural induction on X. In each case, we just show how to establish the first implication; the others are similar.

Case X atomic. We have $W \vdash X \equiv_{AD} W' \vdash Y$ by assumption, so by Lemma 3.9 we have $W \vdash A \equiv_{AD} (W' \vdash Y)[A/X]$. However, since $W' \lhd W$ by assumption, W' does not contain the indicated X, so we have $W \vdash A \equiv_{AD} W' \vdash Y[A/X]$, and are done by taking $U = Y[A/X]$.

Case $X = \sharp X'$. Using the case assumption we have

$$X' \vdash \sharp W \equiv_D W \vdash \sharp X' \equiv_{AD} W' \vdash Y$$

Now since $W' \lhd W$ it is also the case that $W' \lhd \sharp W$, so by part 2 of the induction hypothesis we have $A \vdash \sharp W \equiv_{AD} W' \vdash U'$ for some U'. By Lemma 3.9, we then have $\sharp A \vdash \sharp W \equiv_{AD} (W' \vdash U')[\sharp A/A]$. However, since $W' \lhd W$ it follows that W' does not contain the indicated A, so we have

$$W \vdash A \equiv_D \sharp A \vdash \sharp W \equiv_{AD} W' \vdash U'[\sharp A/A]$$

and we are done by taking $U = U'[\sharp A/A]$.

Case $X = X_1; X_2$. Using the case assumption we have

$$W; \sharp X_2 \vdash X_1 \equiv_D W \vdash X_1; X_2 \equiv_{AD} W' \vdash Y$$

Since $W' \lhd W$ we also have $W' \lhd (W; \sharp X_2)$, so we have by part 1 of the induction hypothesis that $W; \sharp X_2 \vdash A \equiv_{AD} W' \vdash U'$ for some U'. Thus we have $W; \sharp A \vdash X_2 \equiv_{AD} W' \vdash U'$, where $W' \lhd W$ and thus also $W' \lhd (W; \sharp A)$. Using part 1 of the induction hypothesis again we obtain, for some U'',

$$W \vdash A; A \equiv_D W; \sharp A \vdash A \equiv_{AD} W' \vdash U''$$

It must hold that one of the two indicated occurrences of A is delible from $W' \vdash U''$. Furthermore, since $W' \lhd W$, neither indicated A can occur in W'. Thus by Lemma 4.2, we have

$$W \vdash A = (W \vdash A; A) \setminus A \equiv_{AD} (W' \vdash U'') \setminus A = W' \vdash (U'' \setminus A)$$

whence we are done by taking $U = U'' \setminus A$. This completes the proof. □

The essence of the next lemma is that, if one finds that sub-parts of disjoint parts of a consecution have been "mixed together" during an $\rightarrow_{AD}$-rewrite sequence, this must have been achieved with the help of associativity, in which case we can use it to "unmix" those parts.

Lemma 4.6 (Filtration). *Suppose that $X; Y \vdash U \equiv_{AD} W \vdash Z$, where $W \lhd X; Y$ but $W \not\lhd X$ and $W \not\lhd Y$. Then there exist W_1 and W_2 such that $W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z$ with $W_1 \lhd X$ and $W_2 \lhd Y$.*

Similarly, if $X; Y \vdash U \equiv_{AD} Z \vdash W$ with $W \lhd X; Y$ but $W \not\lhd X$ and $W \not\lhd Y$, then there exist W_1 and W_2 such that $Z \vdash W \equiv_{AD} Z \vdash W_1; W_2$ with $W_1 \lhd X$ and $W_2 \lhd Y$.

Proof. We prove both implications simultaneously by induction on the structure of W. We only show how to treat the first implication of the lemma; the second is similar.

Case W atomic. We are done by contradiction because W, being such that $W \lhd X;Y$ but $W \not\lhd X$ and $W \not\lhd Y$ by assumption, must contain at least one semicolon.

Case $W = \sharp W'$. We have $X;Y \vdash U \equiv_{AD} \sharp W' \vdash Z \equiv_{AD} \sharp Z \vdash W'$. Since $\sharp W' \lhd X;Y$ and $\sharp W' \not\lhd X$ and $\sharp W' \not\lhd Y$, it follows that $W' \lhd X;Y$ and $W' \not\lhd X$ and $W' \not\lhd Y$. Thus by (the second part of) the induction hypothesis there exist W_1', W_2' that $\sharp Z \vdash W' \equiv_{AD} \sharp Z \vdash W_1'; W_2'$ and $W_1' \lhd X$ and $W_2' \lhd Y$. Now we proceed as follows:

$$\sharp W' \vdash Z \equiv_D \sharp Z \vdash W' \equiv_{AD} \sharp Z \vdash W_1'; W_2' \equiv_D \sharp W_1'; \sharp W_2' \vdash Z$$

whence we are done by taking $W_1 = \sharp W_1'$ and $W_2 = \sharp W_2'$.

Case $W = W_1; W_2$. If $W_1 \lhd X$ and $W_2 \lhd Y$ or vice versa then we are done. If not, note that by the lemma and case assumptions we have $X;Y \vdash U \equiv_{AD} W_1; W_2 \vdash Z$ where $W_1; W_2 \lhd X;Y$ and either $W_1 \not\lhd X$ and $W_1 \not\lhd Y$, or $W_2 \not\lhd X$ and $W_2 \not\lhd Y$ (or both). It is clear by inspection of the display postulates that this situation can only arise when the associativity rule (A) is present[5]. We show how to treat the case when both W_1 and W_2 contain parts of both X and Y; the cases when only one of W_1, W_2 contains parts of both X and Y are similar.

We have $X;Y \vdash U \equiv_{AD} W_1; W_2 \vdash Z \equiv_D W_1 \vdash \sharp W_2; Z$ and $W_1 \lhd X;Y$ by the case assumption, plus $W_1 \not\lhd X$ and $W_1 \not\lhd Y$ by the subcase assumption above. Thus by (the first part of) the induction hypothesis there exist W_1', W_1'' with $W_1' \lhd X$ and $W_1'' \lhd Y$ such that

$$W_1 \vdash \sharp W_2; Z \equiv_{AD} W_1'; W_1'' \vdash \sharp W_2; Z \equiv_D W_2 \vdash \sharp(W_1'; W_1''); Z$$

Thus by transitivity we have $X;Y \vdash U \equiv_{AD} W_2 \vdash \sharp(W_1'; W_1''); Z$. Again we have $W_2 \lhd X;Y$ by the case assumption and $W_2 \not\lhd X$ and $W_2 \not\lhd Y$ by the subcase assumption above, so, using (the first part of) the induction hypothesis again, there exist W_2', W_2'' with $W_2' \lhd X$ and $W_2'' \lhd Y$ such that

$$W_2 \vdash \sharp(W_1'; W_1''); Z \equiv_{AD} W_2'; W_2'' \vdash \sharp(W_1'; W_1''); Z \equiv_D (W_1'; W_1''); (W_2'; W_2'') \vdash Z$$

Therefore, using the fact that $\equiv_{AD}$ incorporates the associativity rule, we have

$$W_1; W_2 \vdash Z \equiv_{AD} (W_1'; W_1''); (W_2'; W_2'') \vdash Z \equiv_{AD} (W_1'; W_2'); (W_1''; W_2'') \vdash Z$$

We are done by taking $W_1 = (W_1'; W_2')$ and $W_2 = (W_1''; W_2'')$. This completes the proof. □

Theorem 4.7 (Binary rules). *The rules (&R), ($\vee$L) and ($\rightarrow$L) all have the local AD-interpolation property in any extension of $\mathcal{D}_0$.*

[5] This can be proven formally by an induction on $X;Y \vdash U \equiv_{AD} W_1; W_2 \vdash Z$ under the assumption that $\equiv_{AD}$ is exactly $\equiv_D$.

Proof. We show the case of (&R) here; the rules (∨L) and (→L) are similar.

$$\frac{X \vdash F \quad Y \vdash G}{X\,;Y \vdash F \,\&\, G}\,(\&\mathrm{R})$$

By assumption we have interpolants for all $W_1 \vdash Z_1 \equiv_{AD} X \vdash F$ and for all $W_2 \vdash Z_2 \equiv_{AD} Y \vdash G$, and require to find interpolants for all $W \vdash Z \equiv_{AD} X;Y \vdash F \,\&\, G$.

Since $W \vdash Z \equiv_{AD} X;Y \vdash F \,\&\, G$, the indicated $F \,\&\, G$ occurs either in W or Z. We assume it occurs in Z (the case where it occurs in W is symmetric). Thus we have $W \lhd X;Y$. We distinguish three cases: $W \lhd X$, $W \lhd Y$ and neither of these.

Cases $W \lhd X$ and $W \lhd Y$. We show the case $W \lhd X$; the other case is similar. Using the lemma assumption we have

$$X \vdash \sharp Y; F \,\&\, G \equiv_D X;Y \vdash F \,\&\, G \equiv_{AD} W \vdash Z \tag{1}$$

Thus by part 1 of Lemma 4.5 we have $X \vdash F \equiv_{AD} W \vdash U$ for some U. Let I be the interpolant for $W \vdash U$ given by assumption. We claim I is also an interpolant for $W \vdash Z$.

First we check the variable condition. We have $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(U)$ by assumption. To see that $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$, just observe that, since $X \vdash F \equiv_{AD} W \vdash U$ and $X;Y \vdash F \,\&\, G \equiv_{AD} W \vdash Z$ with $W \lhd X$, it is clearly the case that $\mathcal{V}(U) \subseteq \mathcal{V}(Z)$.

Now we check the provability conditions. We trivially have $W \vdash I$ as required. We must show that $I \vdash Z$ is derivable, given that $I \vdash U$ is derivable. First, note that because $X \vdash F \equiv_{AD} W \vdash U$ and $W \lhd X$, the indicated F does not occur in W, so must instead occur (positively) in U. By the display property (Prop. 2.6) we thus have $I \vdash U \equiv_D V \vdash F$ for some V. Next, using $X \vdash F \equiv_{AD} W \vdash U$, (1) above and Lemma 3.9, we have

$$\begin{aligned} W \vdash Z &\equiv_{AD} X \vdash \sharp Y; F \,\&\, G \\ &= \quad X \vdash F[(\sharp Y; F \,\&\, G)/F] \\ &\equiv_{AD} W \vdash U[(\sharp Y; F \,\&\, G)/F] \end{aligned}$$

Thus by Lemma 4.3 we have, using Lemma 3.9 and $I \vdash U \equiv_{AD} V \vdash F$,

$$\begin{aligned} I \vdash Z &\equiv_{AD} I \vdash U[(\sharp Y; F \,\&\, G)/F] \\ &\equiv_{AD} V \vdash F[(\sharp Y; F \,\&\, G)/F] \\ &= \quad V \vdash \sharp Y; F \,\&\, G \\ &\equiv_D \quad V;Y \vdash F \,\&\, G \end{aligned}$$

Thus we can derive $I \vdash Z$ as follows:

$$\cfrac{\cfrac{\cfrac{\vdots}{I \vdash U}}{V \vdash F}\,(\equiv_D) \qquad \cfrac{\vdots}{Y \vdash G}}{\cfrac{V;Y \vdash F \,\&\, G}{I \vdash Z}\,(\equiv_{AD})}\,(\&\mathrm{R})$$

Case $W \lhd X; Y$ *but* $W \not\lhd X$ *and* $W \not\lhd Y$. Since $X; Y \vdash F \& G \equiv_{AD} W \vdash Z$ by general assumption, we can apply the first part of Lemma 4.6 to obtain W_1 and W_2 such that $W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z$ with $W_1 \lhd X$ and $W_2 \lhd Y$. Thus we have both of the following:

$$X \vdash \sharp Y; F \,\&\, G \equiv_{AD} W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z \equiv_D W_1 \vdash \sharp W_2; Z$$
$$Y \vdash \sharp X; F \,\&\, G \equiv_{AD} W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z \equiv_D W_2 \vdash \sharp W_1; Z$$

where $W_1 \lhd X$ and $W_2 \lhd Y$. Thus by part 1 of Lemma 4.5 we have for some U_1 and U_2

$$X \vdash F \equiv_{AD} W_1 \vdash U_1$$
$$Y \vdash G \equiv_{AD} W_2 \vdash U_2$$

Let I_1, I_2 be the interpolants given by assumption for $W_1 \vdash U_1$ and $W_2 \vdash U_2$ respectively. We claim that the formula $I_1 \,\&\, I_2$ is an interpolant for $W \vdash Z$.

First, we check the variable condition. We must show that $\mathcal{V}(I_1 \,\&\, I_2) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$, given that $\mathcal{V}(I_1) \subseteq \mathcal{V}(W_1) \cap \mathcal{V}(U_1)$ and $\mathcal{V}(I_2) \subseteq \mathcal{V}(W_2) \cap \mathcal{V}(U_2)$. It is clear that $\mathcal{V}(W_1) \subseteq \mathcal{V}(W)$ and $\mathcal{V}(W_2) \subseteq \mathcal{V}(W)$ because $W \vdash Z \equiv_{AD} W_1; W_2 \vdash Z$. Moreover, $\mathcal{V}(U_1) \subseteq \mathcal{V}(Z)$ because we have $X \vdash F \equiv_{AD} W_1 \vdash U_1$ and $X \vdash \sharp Y; F \,\&\, G \equiv_{AD} W_1 \vdash \sharp W_2; Z$ while $W_1 \lhd X$ and $W_2 \lhd Y$. Similarly $\mathcal{V}(U_2) \subseteq \mathcal{V}(Z)$ and thus we have, as required:

$$\begin{aligned}
\mathcal{V}(I_1 \,\&\, I_2) &= \mathcal{V}(I_1) \cup \mathcal{V}(I_2) \\
&\subseteq (\mathcal{V}(W_1) \cap \mathcal{V}(U_1)) \cup (\mathcal{V}(W_2) \cap \mathcal{V}(U_2)) \\
&\subseteq (\mathcal{V}(W) \cap \mathcal{V}(Z)) \cup (\mathcal{V}(W) \cap \mathcal{V}(Z)) \\
&= \mathcal{V}(W) \cap \mathcal{V}(Z)
\end{aligned}$$

Now, we check the provability conditions. First, we show that $W \vdash I_1 \,\&\, I_2$ is provable, given that $W_1 \vdash I_1$ and $W_2 \vdash I_2$ are provable by assumption. Since $W_1; W_2 \vdash Z \equiv_{AD} W \vdash Z$, we have $W_1; W_2 \vdash I_1 \,\&\, I_2 \equiv_{AD} W \vdash I_1 \,\&\, I_2$ by Lemma 4.3. Thus we can derive $W \vdash I_1 \,\&\, I_2$ as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ W_1 \vdash I_1\end{matrix} \qquad \begin{matrix}\vdots \\ W_2 \vdash I_2\end{matrix}}{W_1; W_2 \vdash I_1 \,\&\, I_2}\,(\&\mathrm{R})}{W \vdash I_1 \,\&\, I_2}\,(\equiv_{AD})$$

Finally, we must show that $I_1 \,\&\, I_2 \vdash Z$ is derivable, given that $I_1 \vdash U_1$ and $I_2 \vdash U_2$ are derivable. First, note that because $X \vdash F \equiv_{AD} W_1 \vdash U_1$ and $W_1 \lhd X$, the indicated F must occur (positively) in U_1, and thus $I_1 \vdash U_1 \equiv_D V_1 \vdash F$ for some V_1 by the display property (Prop. 2.6). Similarly, $I_2 \vdash U_2 \equiv_D V_2 \vdash G$ for some V_2. Next, since $X \vdash F \equiv_{AD} W_1 \vdash U_1$ we have by Lemma 3.9

$$\begin{aligned}
W_1 \vdash \sharp W_2; Z &\equiv_{AD} W \vdash Z \\
&\equiv_{AD} X \vdash \sharp Y; F \,\&\, G \\
&\equiv_{AD} W_1 \vdash U_1[(\sharp Y; F \,\&\, G)/F]
\end{aligned}$$

Thus by Lemma 4.3 we have

$$I_1 \vdash \sharp W_2; Z \equiv_{AD} I_1 \vdash U_1[(\sharp Y; F \,\&\, G)/F] \tag{2}$$

Since $I_1 \vdash U_1 \equiv_D V_1 \vdash F$ we have by (2) and Lemma 3.9

$$\begin{aligned} I_1 \vdash \sharp W_2; Z &\equiv_{AD} I_1 \vdash U_1[(\sharp Y; F \,\&\, G)/F] \\ &\equiv_D \; V_1 \vdash \sharp Y; F \,\&\, G \\ &\equiv_D \; V_1; Y \vdash F \,\&\, G \end{aligned} \tag{3}$$

Now, since $Y \vdash G \equiv_{AD} W_2 \vdash U_2$ we obtain by Lemma 3.9 and (3) above

$$\begin{aligned} W_2 \vdash \sharp I_1; Z \; &\equiv_D \; I_1 \vdash \sharp W_2; Z \\ &\equiv_{AD} V_1; Y \vdash F \,\&\, G \\ &\equiv_D \; Y \vdash \sharp V_1; F \,\&\, G \\ &\equiv_{AD} W_2 \vdash U_2[(\sharp V_1; F \,\&\, G)/G] \end{aligned}$$

So by applying Lemma 4.3 once more we have

$$I_2 \vdash \sharp I_1; Z \equiv_{AD} I_2 \vdash U_2[\sharp V_1; F \,\&\, G/G] \tag{4}$$

Now, since $I_2 \vdash U_2 \equiv_D V_2 \vdash G$ we obtain by Lemma 3.9 and (4) above

$$\begin{aligned} I_1; I_2 \vdash Z \; &\equiv_D \; I_2 \vdash \sharp I_1; Z \\ &\equiv_{AD} I_2 \vdash U_2[\sharp V_1; F \,\&\, G/G] \\ &\equiv_{AD} V_2 \vdash \sharp V_1; F \,\&\, G \\ &\equiv_D \; V_1; V_2 \vdash F \,\&\, G \end{aligned}$$

This enables us to derive $I_1 \,\&\, I_2 \vdash Z$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{I_1 \vdash U_1}}{V_1 \vdash F}(\equiv_D) \qquad \dfrac{\dfrac{\vdots}{I_2 \vdash U_2}}{V_2 \vdash G}(\equiv_D)}{\dfrac{V_1; V_2 \vdash F \,\&\, G}{\dfrac{I_1; I_2 \vdash Z}{I_1 \,\&\, I_2 \vdash Z}(\&\mathrm{L})}(\equiv_{AD})}(\&\mathrm{R})$$

This completes the subcase, and the proof. $\square$

Corollary 4.8. *For any $\mathcal{D} \in \{\mathcal{D}_0, \mathcal{D}_0^+, \mathcal{D}_0 + (\mathrm{A}), \mathcal{D}_0^+ + (\mathrm{A})\}$, the proof rules of $\mathcal{D}$ all have the LADI property in (any extension of) $\mathcal{D}$, and thus $\mathcal{D}$ has the interpolation property.*

Proof. Let $\mathcal{D} \in \{\mathcal{D}_0, \mathcal{D}_0^+, \mathcal{D}_0 + (\mathrm{A}), \mathcal{D}_0^+ + (\mathrm{A})\}$. LADI for the proof rules of $\mathcal{D}$ in any extension of $\mathcal{D}$ is given by Prop. 3.10 and Theorem 4.7. Interpolation for $\mathcal{D}$ then follows by Lemma 3.5.

5 Interpolation: structural rules

In this section we examine local AD-interpolation for the structural rules given in Figure 3.

Proposition 5.1 (Unit contraction rules). *The unit left-contraction rule* $(\emptyset\mathrm{C_L})$ *has the LADI property in any extension of* $\mathcal{D}_0 + (\emptyset\mathrm{C_L})$*. Similarly, the rule* $(\emptyset\mathrm{C_R})$ *has the LADI property in any extension of* $\mathcal{D}_0 + (\emptyset\mathrm{C_R})$*.*

Proof. We just show the case of $(\emptyset\mathrm{C_L})$, as the case of $(\emptyset\mathrm{C_R})$ is similar.

$$\frac{\emptyset; X \vdash Y}{X \vdash Y}\ (\emptyset\mathrm{C_L})$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} \emptyset; X \vdash Y$, and we require to construct interpolants for all $W \vdash Z \equiv_{AD} X \vdash Y$. We show by reflexive-transitive induction on $W \vdash Z \equiv_{AD} X \vdash Y$ that one of the following holds:

(a) there is an antecedent part U of $W \vdash Z$ such that $(W \vdash Z)[(\emptyset; U)/U] \equiv_{AD} \emptyset; X \vdash Y$, or;
(b) there is a consequent part U of $W \vdash Z$ such that $(W \vdash Z)[(\sharp\emptyset; U)/U] \equiv_{AD} \emptyset; X \vdash Y$.

In the reflexive case we trivially have (a) by taking $U = X$. In the transitive case we have $W \vdash Z \rightarrow_{AD} X' \vdash Y' \equiv_{AD} \emptyset; X \vdash Y$, whence by induction hypothesis either (a) or (b) holds of $X' \vdash Y'$. We assume that (a) holds (the other case is similar) so that for some antecedent part U of $X' \vdash Y'$ we have $(X' \vdash Y')[(\emptyset; U)/U] \equiv_{AD} \emptyset; X \vdash Y$. We proceed by case analysis on $W \vdash Z \rightarrow_{AD} X' \vdash Y'$.

The main interesting cases are when the substructure occurrence U is decomposed by the rewrite. For example, suppose we have $W \vdash Z_1; Z_2 \rightarrow_{AD} W; \sharp Z_1 \vdash Z_2$, and that U is the indicated occurrence of $W; \sharp Z_1$. In that case, we pick V to be the consequent part Z_2 of $W \vdash Z_1; Z_2$, whence we have as required

$$\begin{array}{rcl} (W \vdash Z)[(\sharp\emptyset; V)/V] & = & W \vdash Z_1; (\sharp\emptyset; Z_2) \\ & \equiv_D & \emptyset; (W; \sharp Z_1) \vdash Z_2 \\ & = & (X' \vdash Y')[(\emptyset; U)/U] \\ & \equiv_{AD} & \emptyset; X \vdash Y \qquad \text{(by IH)} \end{array}$$

The other cases are similar. This completes the induction.

Now, we assume without loss of generality that (a) above holds, with U occurring in Z (the other cases are similar). Let I be the interpolant given by assumption for $W \vdash Z[(\emptyset; U)/U]$. We claim that I is also an interpolant for $W \vdash Z$. Clearly the variable condition is satisfied since $\mathcal{V}(Z[(\emptyset; U)/U]) = \mathcal{V}(Z)$. We trivially have $W \vdash I$ provable by assumption. It just remains to verify that $I \vdash Z$ is provable, given that $I \vdash Z[(\emptyset; U)/U]$ is provable. Note that, since U is assumed an antecedent part of $W \vdash Z$, we have $I \vdash Z \equiv_D U \vdash T$ and $I \vdash Z[(\emptyset; U)/U] \equiv_D \emptyset; U \vdash T$ for some T by the display property (Prop. 2.6).

(It is obvious that one obtains the same T by displaying U in $I \vdash Z$ and by displaying $\emptyset; U$ in $I \vdash Z[(\emptyset; U)/U]$, since the display property does not depend on the internal structure of the substructure occurrence being displayed.) Thus we can derive $I \vdash Z$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{I \vdash Z[(\emptyset; U)/U]}}{\emptyset; U \vdash T}\,(\equiv_D)}{U \vdash T}\,(\emptyset \mathrm{C_L})}{I \vdash Z}\,(\equiv_D)$$

This completes the proof. $\square$

Proposition 5.2 (Unit weakening rules). *The unit left-weakening rule* $(\emptyset \mathrm{W_L})$ *has the LADI property in any extension of* $\mathcal{D}_0 + (\emptyset \mathrm{W_L})$. *Similarly, the rule* $(\emptyset \mathrm{W_R})$ *has the LADI property in any extension of* $\mathcal{D}_0 + (\emptyset \mathrm{W_R})$.

Proof. We just show the case of $(\emptyset \mathrm{W_L})$, as the case of $(\emptyset \mathrm{W_R})$ is similar.

$$\dfrac{X \vdash Y}{\emptyset; X \vdash Y}\,(\emptyset \mathrm{W_L})$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} X \vdash Y$, and require to find interpolants for all $W \vdash Z \equiv_{AD} \emptyset; X \vdash Y$.

First of all, suppose that the indicated $\emptyset$ is not delible from $W \vdash Z$, whence either W or Z is of the form $\sharp^n \emptyset$ for some $n \geq 0$. We suppose $Z = \sharp^n \emptyset$ (the other case is similar) in which case n must be odd. We pick the interpolant for $W \vdash Z$ to be $\neg\top$. The variable condition is trivially satisfied. Note that we have

$$\emptyset; X \vdash Y \equiv_{AD} W \vdash Z = W \vdash \sharp^n \emptyset \equiv_D W \vdash \sharp\emptyset$$

Thus $W \vdash \neg\top$ and $\neg\top \vdash Z = \neg\top \vdash \sharp^n \emptyset$ are provable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{X \vdash Y}}{\emptyset; X \vdash Y}\,(\emptyset \mathrm{W_L})}{\emptyset \vdash \sharp W}\,(\equiv_{AD})}{\top \vdash \sharp W}\,(\top \mathrm{L})}{W \vdash \sharp\top}\,(\equiv_D)}{W \vdash \neg\top}\,(\neg \mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{}{\emptyset \vdash \top}\,(\top \mathrm{R})}{\sharp\top \vdash \sharp^n \emptyset}\,(\equiv_D)}{\neg\top \vdash \sharp^n \emptyset}\,(\neg \mathrm{L})$$

Thus we may assume from now on that the indicated $\emptyset$ is delible from $W \vdash Z$. Thus, by Lemma 4.2, we have that

$$X \vdash Y = (\emptyset; X \vdash Y) \setminus \emptyset \equiv_{AD} (W \vdash Z) \setminus \emptyset$$

Let I be the interpolant given for $(W \vdash Z) \setminus F$ by assumption. We claim that I is also an interpolant for $W \vdash Z$. Without loss of generality, we assume that the indicated $\emptyset$ occurs in Z, so that $(W \vdash Z) \setminus \emptyset = W \vdash (Z \setminus \emptyset)$.

First we check the variable condition. By assumption we have $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z \setminus \emptyset)$. By the definition of deletion (Defn. 4.1) we have $Z \setminus \emptyset = Z[S/(\sharp^n\emptyset; S)]$ for some substructure occurrence $(\emptyset; S)$ in Z and for some $n \geq 0$, and thus $\mathcal{V}(Z \setminus \emptyset) = \mathcal{V}(Z)$. Thus $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required.

Now we check the provability conditions. We have $W \vdash I$ provable by assumption, so it just remains to show that $I \vdash Z$ is provable, given that $I \vdash (Z \setminus \emptyset)$ is provable. We assume that the previously indicated $(\sharp^n\emptyset; S)$ is a negative occurrence in Z, thus an antecedent part of $I \vdash Z$ (the other case is similar). Thus n must be even and by the display property (Prop. 2.6) we have, for some T,

$$\begin{array}{rl} & I \vdash Z \equiv_D \sharp^n\emptyset; S \vdash T \equiv_D \emptyset; S \vdash T \\ \text{and} & I \vdash (Z \setminus \emptyset) = I \vdash Z[S/(\sharp^n\emptyset; S)] \equiv_D S \vdash T \end{array}$$

(Note that we obtain the same structure T by displaying S in $I \vdash Z \setminus \emptyset$ and by displaying $(\sharp^n\emptyset; S)$ in $I \vdash Z$.) Thus we can derive $I \vdash Z$ as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{I \vdash (Z \setminus \emptyset)}}{S \vdash T}\ (\equiv_D)}{\emptyset; S \vdash T}\ (\emptyset\mathrm{W_L})}{I \vdash Z}\ (\equiv_D)$$

This completes the proof. □

Lemma 5.3. *The consecutions $X \vdash \top$ and $\neg\top \vdash X$ are provable in any extension of $\mathcal{D}_0 + \{(\mathrm{W}), (\emptyset\mathrm{C_L})\}$. Similarly, the consecutions $\bot \vdash X$ and $X \vdash \neg\bot$ are provable in any extension of $\mathcal{D}_0 + \{(\mathrm{W}), (\emptyset\mathrm{C_R})\}$.*

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\emptyset \vdash \top}\ (\top\mathrm{R})}{\emptyset; \sharp X \vdash \top}\ (\mathrm{W})}{\emptyset; \sharp\top \vdash X}\ (\equiv_D)}{\sharp\top \vdash X}\ (\emptyset\mathrm{C_L})}{\neg\top \vdash X}\ (\neg\mathrm{L}) \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\bot \vdash \emptyset}\ (\bot\mathrm{L})}{\bot; X \vdash \emptyset}\ (\mathrm{W})}{X \vdash \sharp\bot; \emptyset}\ (\equiv_D)}{X \vdash \sharp\bot}\ (\emptyset\mathrm{C_R})}{X \vdash \neg\bot}\ (\neg\mathrm{R})$$

□

Lemma 5.4 (Advanced deletion). *Suppose that either $X; X' \vdash Y \equiv_{AD} W \vdash Z$ or $X \vdash X'; Y \equiv_{AD} W \vdash Z$, where $W \not\trianglelefteq X'$ and $Z \not\trianglelefteq X'$. Then there are atomic parts $A_1, \ldots, A_n$ of the indicated X' such that*

$$X \vdash Y \equiv_{AD} (\ldots((W \vdash Z) \setminus A_1) \setminus A_2 \ldots) \setminus A_n$$

Proof. By structural induction on X'. In each case we assume that $X; X' \vdash Y \equiv_{AD} W \vdash Z$; the other case is similar.

Case X' atomic. The indicated X' is delible from $W \vdash Z$ because $W \not\trianglelefteq X'$ and $Z \not\trianglelefteq X'$ by the lemma assumption. Thus we have by Lemma 4.2 that

$$X \vdash Y = (X; X' \vdash Y) \setminus X' \equiv_{AD} (W \vdash Z) \setminus X'$$

We are done as, trivially, X' is an atomic part of itself.

Case $X' = \sharp X''$. Using the lemma and case assumptions we have

$$X \vdash X''; Y \equiv_D X; \sharp X'' \vdash Y \equiv_{AD} W \vdash Z$$

where $W \not\trianglelefteq \sharp X''$ and $Z \not\trianglelefteq \sharp X''$, from which it follows that $W \not\trianglelefteq X''$ and $Z \not\trianglelefteq X''$. Thus by the second part of the induction hypothesis there exist atomic parts $A_1, \ldots, A_n$ of the indicated X'' such that

$$X \vdash Y \equiv_{AD} (\ldots((W \vdash Z) \setminus A_1) \setminus A_2 \ldots) \setminus A_n$$

Thus, since $A_1, \ldots, A_n$ are also atomic parts of the indicated $\sharp X''$, we are done.

Case $X' = X_1; X_2$. Using the lemma and case assumptions we have

$$X_1; X_2 \vdash \sharp X; Y \equiv_D X; (X_1; X_2) \vdash Y \equiv_{AD} W \vdash Z$$

where $W \not\trianglelefteq (X_1; X_2)$ and $Z \not\trianglelefteq (X_1; X_2)$, from which it follows that $W \not\trianglelefteq X_2$ and $Z \not\trianglelefteq X_2$. Thus by the induction hypothesis there exist atomic parts $A_1, \ldots, A_m$ of the indicated X_2 such that

$$X; X_1 \vdash Y \equiv_D X_1 \vdash \sharp X; Y \equiv_{AD} (\ldots((W \vdash Z) \setminus A_1) \setminus A_2 \ldots) \setminus A_m = W' \vdash Z'$$

Now since $W \not\trianglelefteq (X_1; X_2)$ and $A_1, \ldots, A_m$ are all parts of X_2, it must be the case that $W' \not\trianglelefteq X_1$. Similarly, $Z' \not\trianglelefteq X_1$. Thus by the induction hypothesis there are atomic parts $B_1, \ldots, B_k$ of the indicated X_1 such that

$$\begin{array}{rcl} X \vdash Y & \equiv_{AD} & (\ldots((W' \vdash Z') \setminus B_1) \ldots) \setminus B_k \\ & = & (\ldots(((\ldots(W \vdash Z) \setminus A_1 \ldots) \setminus A_m) \setminus B_1) \ldots) \setminus B_k \end{array}$$

Since $A_1, \ldots, A_m, B_1, \ldots, B_k$ are all atomic parts of $X_1; X_2$, this completes the case, and the proof. $\square$

Theorem 5.5 (Weakening). *The weakening rule* (W) *has the LADI property in any extension of* $\mathcal{D}_0 + \{(\mathrm{W}), (\emptyset \mathrm{C_L})\}$ *or* $\mathcal{D}_0 + \{(\mathrm{W}), (\emptyset \mathrm{C_R})\}$.

Proof.

$$\frac{X \vdash Y}{X; X' \vdash Y}\ (\mathrm{W})$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} X \vdash Y$, and require to find interpolants for all $W \vdash Z \equiv_{AD} X; X' \vdash Y$. We distinguish three cases: $W \trianglelefteq X'$; $Z \trianglelefteq X'$; and neither of the preceding.

Case $W \lhd X'$. We pick the interpolant I for $W \vdash Z$ to be $\top$ if $(\emptyset C_L)$ is available, and $\neg\bot$ if $(\emptyset C_R)$ is available instead. The variable condition on interpolants is trivially satisfied, and $W \vdash I$ is provable by Lemma 5.3.

It remains to show that $I \vdash Z$ is provable. We have $W \vdash Z \equiv_{AD} X' \vdash \sharp X; Y$ and, because $W \lhd X'$, it is also the case that $(\sharp X; Y) \lhd Z$. Thus, by part 4 of Lemma 4.5, we have $I \vdash Z \equiv_{AD} U \vdash \sharp X; Y \equiv_D X; U \vdash Y$ for some U, whence we can derive $I \vdash Z$ as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ X \vdash Y\end{array}}{X; U \vdash Y}\,(\mathrm{W})}{I \vdash Z}\,(\equiv_{AD})$$

Case $Z \lhd X'$. Symmetric to case (a); we pick the interpolant to be $\neg\top$ when $(\emptyset C_L)$ is present, and $\bot$ when $(\emptyset C_R)$ is present instead.

Case $W \ntriangleleft X'$ *and* $Z \ntriangleleft X'$. Using the main and case assumptions, we obtain by Lemma 5.4 atomic parts $A_1, \ldots, A_n$ of the indicated X' such that

$$X \vdash Y \equiv_{AD} (\ldots(((W \vdash Z) \setminus A_1) \setminus A_2)\ldots) \setminus A_n = W' \vdash Z'$$

By the definition of deletion (Defn. 4.1), this means that there are substructure occurrences $(\sharp^{i_1} A_1; U_1), \ldots, (\sharp^{i_n} A_n; U_n)$ such that

$$W' \vdash Z' = (\ldots(((W \vdash Z)[U_1/(\sharp^{i_1} A_1; U_1)])[U_2/(\sharp^{i_2} A_2; U_2)])\ldots)[U_n/(\sharp^{i_n} A_n; U_n)]$$

This means that there is a partition of the aforementioned substructure occurrences into $(\sharp^{a_1} B_1; S_1), \ldots, (\sharp^{a_j} B_j; S_j)$ and $(\sharp^{b_1} C_1; T_1), \ldots, (\sharp^{b_k} C_k; T_k)$ such that

$$\begin{array}{r}W' = (\ldots((W[S_1/(\sharp^{a_1} B_1; S_1)])[S_2/(\sharp^{a_2} B_2; S_2)])\ldots)[S_j/(\sharp^{a_j} B_j; S_j)]\\ Z' = (\ldots((Z[T_1/(\sharp^{b_1} C_1; T_1)])[T_2/(\sharp^{b_2} C_2; T_2)])\ldots)[T_k/(\sharp^{b_k} C_k; T_k)]\end{array} \tag{5}$$

Now let I be the interpolant for $W' \vdash Z'$ given by assumption. We claim that I is also an interpolant for $W \vdash Z$.

First we check the variable condition. We have $\mathcal{V}(I) \subseteq \mathcal{V}(W') \cap \mathcal{V}(Z')$ by assumption. Using (5) it is clear that $\mathcal{V}(W') \subseteq \mathcal{V}(W)$ since W' is obtained by deleting some parts of W, and similarly $\mathcal{V}(Z') \subseteq \mathcal{V}(Z)$. Thus $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required.

Now we check the provability conditions. We just show that $W \vdash I$ is derivable; deriving $I \vdash Z$ is similar. We proceed by induction on the number j of substructure occurrences deleted from W to obtain W'. In the case $j = 0$ we have $W = W'$ and are done since $W' \vdash I$ is provable by assumption. In the case $j > 0$ we have by induction hypothesis that $W[S_1/(\sharp^{a_1} B_1; S_1)] \vdash I$ is provable. We assume that the indicated occurrence of $(\sharp^{a_1} B_1; S_1)$ is a consequent part of $W \vdash I$; the case when it is an antecedent part is similar. Using the display property (Prop. 2.6) to display the occurrence, we derive $W \vdash I$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{W[S_1/(\sharp^{a_1} B_1; S_1)] \vdash I}}{V_1 \vdash S_1}\,(\equiv_D)}{\dfrac{V_1; \sharp^{a_1+1} B_1 \vdash S_1}{\dfrac{V_1 \vdash \sharp^{a_1} B_1; S_1}{W \vdash I}\,(\equiv_D)}\,(\equiv_D)}\,(\mathrm{W})}{}$$

This completes the case, and the proof. □

Lemma 5.6 (Duplication lemma). *Let $\mathcal{D}$ be a calculus that includes the associativity rule* (A)*. Then if $X \vdash Y \equiv_{AD} \mathcal{C}$ then there exist atomic parts $A_1, \ldots, A_n$ of the indicated X such that*

$$X; X \vdash Y \equiv_{AD} \mathcal{C}[(A_1; A_1)/A_1, \ldots, (A_n; A_n)/A_n]$$

Similarly, if $Y \vdash X \equiv_{AD} \mathcal{C}$ then there exist atomic parts $A_1, \ldots, A_n$ of the indicated X such that

$$Y \vdash X; X \equiv_{AD} \mathcal{C}[(A_1; A_1)/A_1, \ldots, (A_n; A_n)/A_n]$$

Proof. We show both implications simultaneously by structural induction on X. In each case we just show how to establish the first implication; the second is similar.

Case X atomic. By Lemma 3.9 we have as required:

$$X; X \vdash Y = (X \vdash Y)[(X; X)/X] \equiv_{AD} \mathcal{C}[(X; X)/X]$$

Case $X = \sharp X'$. Using the case and lemma assumptions we have

$$\sharp Y \vdash X' \equiv_D \sharp X' \vdash Y \equiv_{AD} \mathcal{C}$$

Thus by the second part of the induction hypothesis we have atomic parts $A_1, \ldots, A_n$ of the indicated X' such that

$$\sharp X'; \sharp X' \vdash Y \equiv_D \sharp Y \vdash X'; X' \equiv_{AD} \mathcal{C}[(A_1; A_1)/A_1, \ldots, (A_n; A_n)/A_n]$$

whence we are immediately done using the fact that $A_1, \ldots, A_n$ are atomic parts of the (first) indicated $\sharp X'$.

Case $X = X_1; X_2$. Using the case and lemma assumptions we have

$$X_1 \vdash \sharp X_2; Y \equiv_D X_1; X_2 \vdash Y \equiv_{AD} \mathcal{C}$$

Thus by (the first part of) the induction hypothesis there exist atomic parts $A_1, \ldots, A_m$ of the indicated X_1 such that

$$X_2 \vdash \sharp(X_1; X_1); Y \equiv_D X_1; X_1 \vdash \sharp X_2; Y \equiv_{AD} \mathcal{C}[(A_1; A_1)/A_1, \ldots, (A_m; A_m)/A_m]$$

Thus by (the first part of) the induction hypothesis again there exist atomic parts $B_1, \ldots, B_k$ of the indicated X_2 such that, using the fact that $\equiv_{AD}$ contains (A),

$$\begin{aligned} & (X_1; X_2); (X_1; X_2) \vdash Y \\ \equiv_{AD} \ & X_2; X_2 \vdash \sharp(X_1; X_1); Y \\ \equiv_{AD} \ & \mathcal{C}[(A_1; A_1)/A_1, \ldots, (A_m; A_m)/A_m, (B_1; B_1)/B_1, \ldots, (B_k; B_k)/B_k] \end{aligned}$$

We are done since $A_1, \ldots, A_m, B_1, \ldots, B_k$ are all atomic substructure occurrences in the (first) indicated $X_1; X_2$. □

We remark that Lemma 5.6 fails in display calculi in which the semicolon is not associative. It also fails in display calculi with more than one family of structural connectives.

Proposition 5.7 (Contraction). *The contraction rule* (C) *has the LADI property in any extension of* $\mathcal{D}_0 +$ (A)*.*

Proof.

$$\frac{X; X \vdash Y}{X \vdash Y}\ (\mathrm{C})$$

By assumption we have interpolants for all $W' \vdash Z' \equiv_{AD} X; X \vdash Y$, and we require to find interpolants for an arbitrary $W \vdash Z \equiv_{AD} X \vdash Y$.

Since $\equiv_{AD}$ contains (A) by assumption, we have by Lemma 5.6 that there exist atomic parts $A_1, \ldots, A_n$ of the X indicated in $X \vdash Y$ such that

$$X; X \vdash Y \equiv_{AD} (W \vdash Z)[(A_1; A_1)/A_1, \ldots, (A_n; A_n)/A_n] = W' \vdash Z'$$

We observe that there exists a partition of $\{A_1, \ldots, A_n\}$ into $\{B_1, \ldots, B_j\}$ and $\{C_1, \ldots, C_k\}$ such that

$$\begin{aligned} W' &= W[(B_1; B_1)/B_1, \ldots, (B_j; B_j)/B_j] \\ Z' &= Z[(C_1; C_1)/C_1, \ldots, (C_k; C_k)/C_k] \end{aligned}$$

Let I be the interpolant for $W' \vdash Z'$ given by assumption. We claim that I is also an interpolant for $W \vdash Z$. First we check the variable condition. We have $\mathcal{V}(I) \subseteq \mathcal{V}(W') \cap \mathcal{V}(Z')$ by assumption and, clearly, $\mathcal{V}(W) = \mathcal{V}(W')$ and $\mathcal{V}(Z) = \mathcal{V}(Z')$, so we trivially have $\mathcal{V}(I) \subseteq \mathcal{V}(W) \cap \mathcal{V}(Z)$ as required.

Finally, we check the provability conditions. We just show that $W \vdash I$ is provable, as the case of $I \vdash Z$ is similar. We proceed by induction on the number j of structures duplicated in W to obtain W'. In the case $j = 0$ we have $W = W'$ and are done since $W' \vdash I$ is provable by assumption. In the case $j > 0$ we have that $W[(B_1; B_1)/B_1] \vdash I$ is provable by induction hypothesis. We assume that

B_1 is a consequent part of W (the case where it is an antecedent part is similar), and derive $W \vdash I$ as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{W[(B_1;B_1)/B_1] \vdash I}}{V_1 \vdash B_1;B_1}\,(\equiv_D)}{\sharp B_1;\sharp B_1 \vdash \sharp V_1}\,(\equiv_D)}{\sharp B_1 \vdash \sharp V_1}\,(\mathrm{C})}{\dfrac{V_1 \vdash B_1}{W \vdash I}\,(\equiv_D)}\,(\equiv_D)
$$

using the display property (Prop. 2.6) to display and then "undisplay" substructure occurrences as appropriate. This completes the induction, and the proof. □

Our conditions on calculi with the contraction rule in Prop. 5.7 can be motivated by a simple example. Suppose that the rule (A) is not present so that $\equiv_{AD}$ is exactly $\equiv_D$, and consider the following instance of contraction.

$$
\frac{(X_1;X_2);(X_1;X_2) \vdash Y}{X_1;X_2 \vdash Y}
$$

In particular, to show the LADI property, we must find an interpolant for $X_1 \vdash \sharp X_2;Y \;\equiv_D\; X_1;X_2 \vdash Y$. However, due to the absence of associativity, we cannot rearrange the premise into $X_1;X_1 \vdash (\sharp X_2;\sharp X_2);Y$ as would otherwise be provided by Lemma 5.6. The best we can do without associativity is $X_1 \vdash \sharp X_2;(\sharp(X_1;X_2);Y)$, an interpolant I for which is too weak to serve as an interpolant for $X_1 \vdash \sharp X_2;Y$ both in terms of provability and in terms of the variable condition. A similar problem occurs if there is more than one binary structural connective, even if both are associative.

The dependencies for local AD-interpolation of the various proof rules are set out in Figure 4. As a result, we have the following interpolation results.

Theorem 5.8 (Interpolation). *Let $\mathcal{D}$ be an extension of $\mathcal{D}_0$ subject to the constraint that if $\mathcal{D}$ contains* (C) *it must also contain* (A)*, and if $\mathcal{D}$ contains* (W) *then it must also contain either* ($\emptyset\mathrm{C_L}$) *or* ($\emptyset\mathrm{C_R}$)*. Then $\mathcal{D}$ has the interpolation property.*

Proof. By Lemma 3.5 it suffices to prove local AD-interpolation in $\mathcal{D}$ for each of the proof rules of $\mathcal{D}$. The rules of $\mathcal{D}_0$, and (A) if applicable, satisfy local AD-interpolation in $\mathcal{D}$ by Corollary 4.8. The other structural rules of $\mathcal{D}$, if applicable, satisfy local AD-interpolation in $\mathcal{D}$ by Theorem 5.5 and Propositions 5.1, 5.2 and 5.7.

In particular, drawing on the observations in Comment 2.8, Theorem 5.8 yields the following:

Corollary 5.9. *$\mathcal{D}_{\mathrm{MLL}}$, $\mathcal{D}_{\mathrm{MALL}}$ and $\mathcal{D}_{\mathrm{CL}}$ have the interpolation property.*

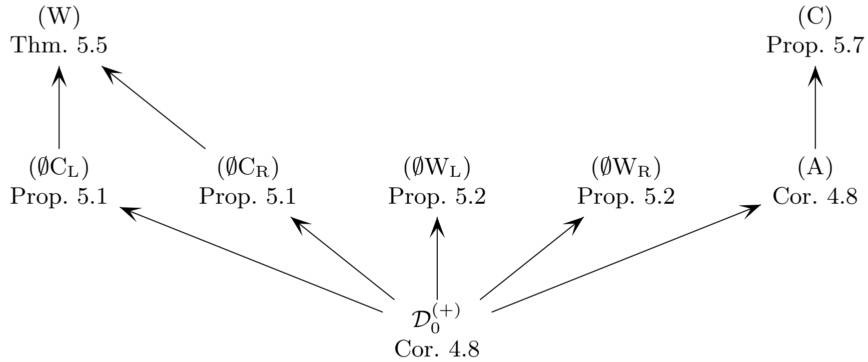

Fig. 4. Diagrammatic summary of our results. Local AD-interpolation of the proof rule(s) at a node holds in a calculus with all of the proof rules at its ancestor nodes.

6 Related and future work

The central contribution of the present paper is a general proof-theoretic method for establishing Craig interpolation in displayable logics, based upon an analysis of the individual rules of their display calculi. This analysis is "as local as possible" in that the required local AD-interpolation property to be satisfied by each proof rule typically depends only on the presence of certain other rules in the calculus, and the syntax of the rule itself. The practicality and generality of our method is demonstrated here by its application to a fairly large family of display calculi differing in their structural rules (and the presence of additive logical connectives). We obtain by this uniform method the interpolation property for MLL, MALL and ordinary classical logic, as well as numerous variants of these logics. To our knowledge, ours are the first interpolation results to be based on display calculi. Thus, in particular, we provide a positive response to Belnap's long-standing open question (cf. [1]) of whether display logic can be used as a basis for establishing an interpolation result.

While interpolation based on display calculi appears to be new, interpolation for substructural logics has of course been considered before. The closest work to our own is probably that of Roorda [17] who demonstrates interpolation for various fragments of classical linear logic, using induction over cut-free sequent calculus proofs, and identifies fragments in which interpolation fails (because certain logical connectives are unavailable). Many of Roorda's positive interpolation results overlap with our own. However, compared to this work, we cover some additional logics (e.g., full classical logic, nonassociative or affine logics) and offer an analysis of the roles played by individual structural rules. An entirely different approach to interpolation for substructural logics is offered by Galatos and Ono [8], who establish very general interpolation theorems for certain substructural logics obtained as extensions of the Lambek calculus, based on their algebraisations.

We remark that our methodology transfers easily to calculi for intuitionistic-type logics in which our "classical" display postulates in Defn. 2.5 are replaced by “residuated” display postulates:

$$X, Y \vdash Z \rightleftarrows_D X \vdash Y, Z \rightleftarrows_D Y, X \vdash Z$$

(where the comma is interpreted as conjunction in antecedent position and as implication in consequent position). A more challenging technical extension is to the case where we have such a family of structural connectives *alongside* the first, as is needed to display relevant logics [16] or bunched logics [2]. Here, the main technical obstacle is in extending the crucial substitutivity principles in Section 4 to the more complex notion of display-equivalence induced by this extension. Other possible extensions to our calculi include the addition of modalities, quantifiers or linear exponentials. In the main, these extensions appear more straightforward than adding new connective families, since they necessitate little or no modification to display-equivalence. We also note that our notion of interpolant in this paper is relatively blunt since it does not distinguish between positive and negative occurrences of propositional variables. It should be possible to read off a sharpened version of interpolation, that does make this distinction, more or less directly from our proof as written.

As well as showing interpolation for a variety of substructural logics, our proof gives insights into the reasons why interpolation fails in some logics. Specifically, we identify contraction as being just as problematic for interpolation as it typically is for decidability (and even weakening causes an issue for interpolation when the logic lacks strong units). Our interpolation method is bound to fail for any multiple-family display calculus including a contraction rule, due to our observation that contraction generally has the required LADI property only in certain circumstances which are precluded by the presence of multiple binary structural connectives. This observation is in keeping with the fact that interpolation fails for the relevant logic **R**, as famously shown by Urquhart [18], since its display calculus employs two families of connectives and a contraction rule. We conjecture that interpolation fails in bunched logics such as BI for similar reasons.

The technical overhead of our method is fairly substantial, but the techniques themselves are elementary: we mainly appeal to structural induction on structures and/or reflexive-transitive induction on equivalence relations such as $\equiv_{AD}$. The elementary nature of the reasoning combined with the proliferation of cases means that our proofs are good candidates for mechanisation in a theorem proving assistant. Our colleague Jeremy Dawson is currently working on an Isabelle formalisation of our proofs, based upon earlier work on mechanising display calculus with Goré in [6]. We note that our interpolation proofs here are fully constructive so, in principle, one can use them to extract the interpolant from a given cut-free display calculus proof. Thus, as well as providing the greatest possible degree of confidence in our proofs, such a mechanisation might also eventually serve as the basis for an automated interpolation tool. Finally, we note that it should also be possible to extract bounds on the size of the interpolant for a consecution, given bounds on the size of its cut-free proof.

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